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Abstract

Let I be an open interval containing 0, M a real manifold, $\dot{T}^*M$ its cotangent bundle with the zero-section removed and $\Phi = \{\varphi_t\}_{t\in I}$ a homogeneous Hamiltonian isotopy of $\dot{T}^*M$ with $\varphi_0 = \mathrm{id}$. Let $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ be the conic Lagrangian submanifold associated with Φ. We prove the existence and unicity of a sheaf K on $M \times M \times I$ whose microsupport is contained in the union of Λ and the zero-section and whose restriction to $t = 0$ is the constant sheaf on the diagonal of $M \times M$. We give applications of this result to problems of non-displaceability in contact and symplectic topology. In particular we prove that some strong Morse inequalities are stable by Hamiltonian isotopies and we also give results of non-displaceability for non-negative isotopies in the contact setting.

Introduction

The microlocal theory of sheaves has been introduced and systematically developed in [11, 12, 13], the central idea being that of the microsupport of sheaves. More precisely, consider a real manifold M of class C^∞ and a commutative unital ring $\mathbf{k}$ of finite global dimension. Denote by $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M. The microsupport

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$\mathrm{SS}(F)$ of an object F of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ is a closed subset of the cotangent bundle T^*M, conic for the action of $\mathbb{R}^+$ on T^*M and co-isotropic. Hence, this theory is "conic", that is, it is invariant by the $\mathbb{R}^+$-action and is related to the homogeneous symplectic structure rather than the symplectic structure of T^*M.

In order to treat non-homogeneous symplectic problems, a classical trick is to add a variable which replaces the homogeneity. This is performed for complex symplectic manifolds in [22] and later in the real case by D. Tamarkin in [24] who adapts the microlocal theory of sheaves to the non-homogeneous situation and deduces a new and very original proof of the classical non-displaceability theorem conjectured by Arnold. (Tamarkin's ideas have also been exposed and developed in [9].) Note that the use of sheaf theory in symplectic topology already appeared in [15], [19], [20] and [21].

In this paper, we will also find a new proof of the non-displaceability theorem and other related results, still remaining in the homogeneous symplectic framework, which makes the use of sheaf theory much easier. In other words, instead of adapting microlocal sheaf theory to treat non-homogeneous geometrical problems, we translate these geometrical problems to homogeneous ones and apply the classical microlocal sheaf theory. Note that the converse is not always possible: there are interesting geometrical problems, for example those related to the notion of non-negative Hamiltonian isotopies, which make sense in the homogeneous case and which have no counterpart in the purely symplectic case.

Our main tool is, as seen in the title of this paper, a quantization of Hamiltonian isotopies in the category of sheaves. More precisely, we consider a homogeneous Hamiltonian isotopy $\Phi = \{\varphi_t\}_{t\in I}$ of $\dot{T}^*M$ (the complementary of the zero-section of T^*M) defined on an open interval I of $\mathbb{R}$ containing 0 such that $\varphi_0 = \mathrm{id}$. Denoting by $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ the conic Lagrangian submanifold associated with Φ, we prove that there exists a unique $K \in \mathsf{D}(\mathbf{k}_{M\times M\times I})$ whose microsupport is contained in the union of Λ and the zero-section of $T^*(M \times M \times I)$ and whose restriction to $t = 0$ is the constant sheaf on the diagonal of $M \times M$.

We give a few applications of this result to problems of non-displaceability in symplectic and contact geometry. The classical non-displaceability conjecture of Arnold says that, on the cotangent bundle to a compact manifold M, the image of the zero-section of T^*M by a Hamiltonian isotopy always intersects the zero-section. This conjecture (and its refinements, using Morse inequalities) have been proved by Chaperon [1] who treated the case of the torus using the methods of Conley and Zehnder [6], then by Hofer [10] and Laudenbach and Sikorav[17]. For related results in the contact case, let us quote in particular Chaperon [2], Chekanov [3] and Ferrand [8].

In this paper we recover the non-displaceability result in the symplectic case as well as its refinement using Morse inequalities. Indeed, we deduce these results from their homogeneous counterparts which are easy corollaries of our theorem of quantization of homogeneous Hamiltonian isotopies. We also study non-negative Hamiltonian isotopies (which make sense only in the contact setting): we prove that given two compact connected submanifolds X and Y in a connected non-compact manifold M and a non-negative Hamiltonian isotopy $\Phi = \{\varphi_t\}_{t\in I}$ such that φ_{t_0} interchanges the conormal bundle to X with that of Y, then $X = Y$ and φ_t induces the identity on the conormal bundle to X for $t \in [0, t_0]$. The first part of these results has already been obtained when X and Y are points in [4, 5].

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1 Microlocal theory of sheaves, after [13]

In this section, we recall some definitions and results from [13], following its notations with the exception of slight modifications. We consider a real manifold M of class C^∞.

1.1 Some geometrical notions ([13, §4.2, §6.2])

In this paper we say that a C^1-map $f\colon M \to N$ is *smooth* if its differential $d_x f\colon T_x M \to T_{f(x)}N$ is surjective for any $x \in M$. For a locally closed subset A of M, one denotes by $\mathrm{Int}(A)$ its interior and by $\overline{A}$ its closure. One denotes by Δ_M or simply Δ the diagonal of $M \times M$.

One denotes by $\tau_M\colon TM \to M$ and $\pi_M\colon T^*M \to M$ the tangent and cotangent bundles to M. If $L \subset M$ is a submanifold, one denotes by $T_L M$ its normal bundle and $T^*_L M$ its conormal bundle. They are defined by the exact sequences

$$\begin{aligned}&0 \to TL \to L \times_M TM \to T_L M \to 0,\\&0 \to T^*_L M \to L \times_M T^*M \to T^*L \to 0.\end{aligned}$$

One sometimes identifies M with the zero-section $T^*_M M$ of T^*M. One sets $\dot{T}^*M := T^*M \setminus T^*_M M$ and one denotes by $\dot{\pi}_M \colon \dot{T}^*M \to M$ the projection.

Let $f\colon M \to N$ be a morphism of real manifolds. To f are associated the tangent morphisms

$$\begin{tikzcd} TM \arrow[r, "f'"] \arrow[d, "\tau_M"] & M \times_N TN \arrow[r, "f_\tau"] \arrow[d] & TN \arrow[d, "\tau_N"] \\ M \arrow[r, equal] & M \arrow[r, "f"] & N. \end{tikzcd} \tag{1.1}$$

By duality, we deduce the diagram:

$$\begin{tikzcd} T^*M & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d] & T^*N \arrow[d, "\pi_N"] \\ M \arrow[u, leftarrow, "\pi_M"] \arrow[r, equal] & M \arrow[r, "f"] & N. \end{tikzcd} \tag{1.2}$$

One sets

$$T^*_M N := \operatorname{Ker} f_d = f_d^{-1}(T^*_M M) \subset M \times_N T^*N.$$

Note that, denoting by Γ_f the graph of f in $M \times N$, the projection $T^*(M \times N) \to M \times T^*N$ identifies $T^*_{\Gamma_f}(M \times N)$ and $M \times_N T^*N$.

Now consider the homogeneous symplectic manifold T^*M: it is endowed with the Liouville 1-form given in a local homogeneous symplectic coordinate system $(x;\xi)$ on T^*M by

$$\alpha_M = \langle \xi, dx \rangle.$$

The antipodal map a_M is defined by:

$$a_M \colon T^*M \to T^*M, \quad (x;\xi) \mapsto (x;-\xi). \tag{1.3}$$

If A is a subset of T^*M, we denote by A^a instead of $a_M(A)$ its image by the antipodal map.

We shall use the Hamiltonian isomorphism $H\colon T^*(T^*M) \xrightarrow{\sim} T(T^*M)$ given in a local symplectic coordinate system $(x;\xi)$ by

$$H(\langle \lambda, dx \rangle + \langle \mu, d\xi \rangle) = -\langle \lambda, \partial_\xi \rangle + \langle \mu, \partial_x \rangle.$$

1.2 Microsupport ([13, §5.1, §6.5])

We consider a commutative unital ring $\mathbf{k}$ of finite global dimension (*e.g.* $\mathbf{k} = \mathbb{Z}$). (We shall assume that $\mathbf{k}$ is a field when we use Morse inequalities in section 4.) We denote by $\mathsf{D}(\mathbf{k}_M)$ (resp. $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$) the derived category (resp. bounded derived category) of sheaves of $\mathbf{k}$-modules on M. We denote by $\omega_M \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ the dualizing complex on M. Recall that ω_M is isomorphic to the orientation sheaf shifted by the dimension. We shall recall the definition of the microsupport (or singular support) $\mathrm{SS}(F)$ of a sheaf F ([13, Def. 5.1.2]).

Definition 1.1. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in U$, one has $\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}(F)_{x_0} \simeq 0$.

In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- By its construction, the microsupport is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$.
- The microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

In the sequel, for a locally closed subset Z of M, we denote by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z, extended by 0 on $M \setminus Z$.

Example 1.2. (i) If F is a non-zero local system on M and M is connected, then $\mathrm{SS}(F) = T^*_M M$.
(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.
(iii) Let φ be a C^1-function such that $d\varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\begin{aligned}
\mathrm{SS}(\mathbf{k}_U) &= U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},\\
\mathrm{SS}(\mathbf{k}_Z) &= Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.
\end{aligned}$$

For a precise definition of being involutive (or co-isotropic), we refer to [13, Def. 6.5.1]

Theorem 1.3. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Then its microsupport $\mathrm{SS}(F)$ is involutive.*

1.3 Localization ([13, §6.1])

Now let A be a subset of T^*M and let $Z = T^*M \setminus A$. The full subcategory $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$ of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ consisting of sheaves F such that $\mathrm{SS}(F) \subset Z$ is triangulated. One sets

$$\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A) := \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)/\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M),$$

the localization of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ by $\mathsf{D}^{\mathrm{b}}_Z(\mathbf{k}_M)$. Hence, the objects of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ are those of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ but a morphism $u\colon F_1 \to F_2$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ becomes an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; A)$ if, after embedding this morphism in a distinguished triangle $F_1 \to F_2 \to F_3 \xrightarrow{+1}$, one has $\mathrm{SS}(F_3) \cap A = \emptyset$.

When $A = \{p\}$ for some $p \in T^*M$, one simply writes $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; p)$ instead of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; \{p\})$.

1.4 Functorial operations ([13, §5.4])

Let M and N be two real manifolds. We denote by q_i ($i = 1, 2$) the i-th projection defined on $M \times N$ and by p_i ($i = 1, 2$) the i-th projection defined on $T^*(M \times N) \simeq T^*M \times T^*N$.

Definition 1.4. Let $f\colon M \to N$ be a morphism of manifolds and let $\Lambda \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. One says that f is non-characteristic for Λ (or else, Λ is non-characteristic for f) if

$$f_\pi^{-1}(\Lambda) \cap T^*_M N \subset M \times_N T^*_N N.$$

If Λ is a closed $\mathbb{R}^+$-conic subset of $\dot{T}^*N$, we say that Λ is non-characteristic for f if so is $\Lambda \cup T^*_N N$.

A morphism $f\colon M \to N$ is non-characteristic for a closed $\mathbb{R}^+$-conic subset Λ if and only if $f_d\colon M \times_N T^*N \to T^*M$ is proper on $f_\pi^{-1}(\Lambda)$ and in this case $f_d f_\pi^{-1}(\Lambda)$ is closed and $\mathbb{R}^+$-conic in T^*M.

Theorem 1.5. (See [13, §5.4].) Let $f\colon M \to N$ be a morphism of manifolds, let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$.

(i) Assume that f is proper on $\mathrm{Supp}(F)$. Then $\mathrm{SS}(\mathrm{R}f_!F) \subset f_\pi f_d^{-1}\mathrm{SS}(F)$.

(ii) Assume that f is non-characteristic for $\mathrm{SS}(G)$. Then $\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}\mathrm{SS}(G)$.

(iii) Assume that f is smooth. Then $\mathrm{SS}(F) \subset M \times_N T^*N$ if and only if, for any $j \in \mathbb{Z}$, the sheaves $H^j(F)$ are locally constant on the fibers of f.

There exist estimates of the microsupport for characteristic inverse images and (in some special situations) for non-proper direct images but we shall not use them here.

Corollary 1.6. *Let I be a contractible manifold and let $p\colon M \times I \to M$ be the projection. If $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I})$ satisfies $\mathrm{SS}(F) \subset T^*M \times T^*_I I$, then $F \simeq p^{-1}\mathrm{R}p_*F$.*

Proof. The result follows from Theorem 1.5 (iii) and [13, Prop. 2.7.8]. Q.E.D.

Corollary 1.7. *Let I be an open interval of $\mathbb{R}$ and let $q\colon M \times I \to I$ be the projection. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I})$ such that $\mathrm{SS}(F) \cap T^*_M M \times T^*I \subset T^*_{M\times I}(M \times I)$ and q is proper on $\mathrm{Supp}(F)$. Then, setting $F_a := F|_{\{t=a\}}$, we have isomorphisms $\mathrm{R}\Gamma(M; F_s) \simeq \mathrm{R}\Gamma(M; F_t)$ for any $s, t \in I$.*

Proof. It follows from Theorem 1.5 that $\mathrm{SS}(\mathrm{R}q_*F) \subset T^*_I I$. Therefore, there exists $M \in \mathsf{D}^{\mathrm{b}}(\mathbf{k})$ and an isomorphism $\mathrm{R}q_*F \simeq M_I$. (Recall that $M_I = a_I^{-1}M$, where $a_I \to \{\mathrm{pt}\}$ is the projection and M is identified to a sheaf on $\{\mathrm{pt}\}$.) Since $\mathrm{R}\Gamma(M; F_s) \simeq (\mathrm{R}q_*F)_s$, the result follows. Q.E.D.

1.5 Morse Lemma and Morse inequalities ([13, §5.4])

In this subsection, we consider a function $\psi\colon M \to \mathbb{R}$ of class C^1. We set

$$\Lambda_\psi = \{(x; d\psi(x))\} \subset T^*M. \tag{1.4}$$

The proposition below is a particular case of the microlocal Morse lemma (see [13, Cor. 5.4.19]) and follows from Theorem 1.5 (ii). The classical theory corresponds to the constant sheaf $F = \mathbf{k}_M$.

Proposition 1.8. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$, let $\psi\colon M \to \mathbb{R}$ be a function of class C^1 and assume that ψ is proper on $\mathrm{Supp}(F)$. Let $a < b$ in $\mathbb{R}$ and assume that $d\psi(x) \notin \mathrm{SS}(F)$ for $a \leq \psi(x) < b$. For $t \in \mathbb{R}$, set $M_t = \psi^{-1}(]-\infty, t[)$. Then the restriction morphism $\mathrm{R}\Gamma(M_b; F) \to \mathrm{R}\Gamma(M_a; F)$ is an isomorphism.*

Corollary 1.9. *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $\psi\colon M \to \mathbb{R}$ be a function of class C^1. Let Λ_ψ be as in (1.4). Assume that*

(i) $\mathrm{Supp}(F)$ *is compact,*

(ii) $\mathrm{R}\Gamma(M;F) \neq 0$.

Then $\Lambda_\psi \cap \mathrm{SS}(F) \neq \emptyset$.

Until the end of this subsection as well as in Section 4.4 we assume that **k** is a field. The classical Morse inequalities are extended to sheaves (see [23] and [13, Prop. 5.4.20]). Let us briefly recall this result.

For a bounded complex E of **k**-vector spaces with finite-dimensional cohomologies, we set

$$b_j(E) = \dim H^j(E), \quad b^*_l(E) = (-1)^l \sum_{j \le l} (-1)^j b_j(E).$$

We consider a map $\psi \colon M \to \mathbb{R}$ of class C^1 and define Λ_ψ as above. Note that we do not ask ψ to be smooth. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. Assume that

$$\text{the set } \Lambda_\psi \cap \mathrm{SS}(F) \text{ is finite, say } \{p_1, \dots, p_N\} \tag{1.5}$$

and, setting

$$x_i = \pi(p_i), \quad V_i := (\mathrm{R}\Gamma_{\{\psi(x) \ge \psi(x_i)\}}(F))_{x_i}, \tag{1.6}$$

also assume that

(1.7) for all $i \in \{1, \dots, N\}, j \in \mathbb{Z}$, the spaces $H^j(V_i)$ are finite-dimensional.

Set

$$b_j(F) = b_j(\mathrm{R}\Gamma(M;F)), \quad b^*_j(F) = b^*_j(\mathrm{R}\Gamma(M;F)).$$

Theorem 1.10. *Let* $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ *and assume that* ψ *is proper on* $\mathrm{Supp}(F)$. *Assume moreover* (1.5) *and* (1.7). *Then*

$$b^*_l(F) \le \sum_{i=1}^N b^*_l(V_i) \text{ for any } l. \tag{1.8}$$

In fact the assumption that ψ is proper on $\mathrm{Supp}(F)$ may be weakened, see loc. cit.

Notice that (1.8) immediately implies

$$b_j(F) \le \sum_{i=1}^N b_j(V_i) \quad \text{for any } j. \tag{1.9}$$

1.6 Kernels ([13, §3.6])

Let M_i ($i = 1, 2, 3$) be manifolds. For short, we write $M_{ij} := M_i \times M_j$ ($1 \le i, j \le 3$) and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. Similarly, we denote by p_i the projection $T^*M_{ij} \to T^*M_i$ or the projection $T^*M_{123} \to T^*M_i$ and by p_{ij} the projection $T^*M_{123} \to T^*M_{ij}$. We also need to introduce the map p_{12^a}, the composition of p_{12} and the antipodal map on T^*M_2.

Let $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$. We set

$$\Lambda_1 \circ \Lambda_2 := p_{13}({p_{12^a}}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2). \tag{1.10}$$

We consider the operation of convolution of kernels:

$$\begin{aligned}\underset{M_2}{\circ} : \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) &\to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\ (K_1, K_2) &\mapsto K_1 \underset{M_2}{\circ} K_2 := \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23}^{-1}K_2).\end{aligned}$$

Let $\Lambda_i = \mathrm{SS}(K_i) \subset T^*M_{i,i+1}$ and assume that

$$\begin{cases} \text{(i) } q_{13} \text{ is proper on } q_{12}^{-1}\operatorname{Supp}(K_1) \cap q_{23}^{-1}\operatorname{Supp}(K_2), \\ \text{(ii) } \ p_{12^a}^{-1}\Lambda_1 \cap p_{23}^{-1}\Lambda_2 \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3) \\ \qquad\qquad \subset T^*_{M_1\times M_2\times M_3}(M_1 \times M_2 \times M_3). \end{cases} \tag{1.11}$$

It follows from Theorem 1.5 that under the assumption (1.11) we have:

$$\mathrm{SS}(K_1 \underset{M_2}{\circ} K_2) \subset \Lambda_1 \circ \Lambda_2. \tag{1.12}$$

If there is no risk of confusion, we write $\circ$ instead of $\underset{M_2}{\circ}$.

We will also use a relative version of the convolution of kernels. For a manifold I, $K_1 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}\times I})$ and $K_2 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}\times I})$ we set

$$K_1 \circ|_I K_2 := \mathrm{R}q_{13I!}(q_{12I}^{-1}K_1 \overset{\mathrm{L}}{\otimes} q_{23I}^{-1}K_2), \tag{1.13}$$

where q_{ijI} is the projection $M_{123} \times I \to M_{ij} \times I$. The above results extend to the relative case. Namely, we assume the conditions:

$$\begin{cases} \text{(i) } \operatorname{Supp}(K_1) \times_{M_2\times I} \operatorname{Supp}(K_2) \longrightarrow M_{13} \times I \text{ is proper,} \\ \text{(ii) } \ p_{12^aI^a}^{-1}\Lambda_1 \cap p_{23I}^{-1}\Lambda_2 \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3 \times T^*I) \\ \qquad\qquad \subset T^*_{M_1\times M_2\times M_3\times I}(M_1 \times M_2 \times M_3 \times I), \end{cases} \tag{1.14}$$

where $p_{12^aI^a}\colon T^*M_1\times T^*M_2\times T^*M_3\times T^*I \longrightarrow T^*M_1\times T^*M_2\times T^*I$ is given by $p_{12^aI^a}(v_1,v_2,v_3,u)=(v_1,-v_2,-u)$. Then we have

$$\mathrm{SS}(K_1\circ|_I K_2) \quad\subset\quad \Lambda_1\circ|_I\Lambda_2 := r_{13}(r_{12^a}^{-1}\Lambda_1\cap r_{23}^{-1}\Lambda_2). \tag{1.15}$$

Here, in the diagram

$$\begin{tikzcd}[column sep=tiny]
 & T^*M_1\times T^*M_2\times T^*M_3\times(T^*I\times_I T^*I) \arrow[dl, "r_{12^a}"'] \arrow[d, "r_{13}"'] \arrow[dr, "r_{23}"] & \\
T^*M_1\times T^*M_2\times T^*I & T^*M_1\times T^*M_3\times T^*I & T^*M_2\times T^*M_3\times T^*I,
\end{tikzcd}$$

r_{12^a} is given by $p_{12^a}\colon T^*M_1\times T^*M_2\times T^*M_3\to T^*M_1\times T^*M_2$ and the first projection $T^*I\times_I T^*I\to T^*I$, r_{23} is given by $p_{23}\colon T^*M_1\times T^*M_2\times T^*M_3\to T^*M_2\times T^*M_3$ and the second projection $T^*I\times_I T^*I\to T^*I$, and r_{13} is given by $p_{13}\colon T^*M_1\times T^*M_2\times T^*M_3\to T^*M_1\times T^*M_3$ and the addition map $T^*I\times_I T^*I\to T^*I$.

1.7 Locally bounded categories and gluing sheaves

Although the prestack $U\mapsto \mathsf{D}(\mathbf{k}_U)$ (U open in M) is not a stack, we have the following classical result that we shall use.

Lemma 1.11. *Let U_1 and U_2 be two open subsets of M and set $U_{12}{:=}U_1\cap U_2$. Let $F_i\in\mathsf{D}(\mathbf{k}_{U_i})$ $(i=1,2)$ and assume we have an isomorphism $\varphi_{21}\colon F_1|_{U_{12}}\simeq F_2|_{U_{12}}$. Then there exists $F\in\mathsf{D}(\mathbf{k}_{U_1\cup U_2})$ and isomorphisms $\varphi_i\colon F|_{U_i}\simeq F_i$ $(i=1,2)$ such that $\varphi_{12}=\varphi_2|_{U_{12}}\circ\varphi_1|_{U_{12}}^{-1}$. Moreover, such a triple (F,φ_1,φ_2) is unique up to a (non-unique) isomorphism.*

Proof. (i) For $*=1,2$ or 12 we let j_* be the inclusion of U_* in $U_1\cup U_2$. By adjunction between $j_{12!}$ and j_{12}^{-1}, the morphism φ_{21} defines the morphism $u\colon j_{12!}(F_1|_{U_1})\to j_{2!}F_2$. We also have the natural morphism $v\colon j_{12!}(F_1|_{U_1})\to j_{1!}F_1$. Then F is given by the distinguished triangle

$$j_{12!}(F_1|_{U_1})\xrightarrow{u\oplus v} j_{2!}F_2\oplus j_{1!}F_1\to F\xrightarrow{+1}.$$

(ii) The unicity follows from the distinguished triangle $F_{U_{12}}\to F_{U_1}\oplus F_{U_2}\to F\xrightarrow{+1}$ and the fact that a commutative square in $\mathsf{D}(\mathbf{k}_M)$ can be extended to a morphism of distinguished triangles. Q.E.D.

Definition 1.12. We say that $F \in \mathsf{D}(\mathbf{k}_M)$ is locally bounded if for any relatively compact open subset $U \subset M$ we have $F|_U \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U)$. We denote by $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ be the full subcategory of $\mathsf{D}(\mathbf{k}_M)$ consisting of locally bounded objects.

Local notions defined for objects of $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ extend to objects of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$, in particular the microsupport. The Grothendieck operations which preserve boundedness properties also preserve the local boundedness except maybe direct images. However for $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ and a morphism of manifolds $f\colon M \to N$ which is proper on $\mathrm{Supp}(F)$ we have $\mathrm{R}f_*F \simeq \mathrm{R}f_!F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_N)$ and Theorem 1.5 still holds in this context.

In particular in the situation of the previous paragraph if we assume that $K_1 \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M_{12}})$ and $K_2 \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M_{23}})$ satisfy (1.11), then we obtain $K_1 \underset{M_2}{\circ} K_2 \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M_{13}})$ with the same bound for its microsupport.

The category $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ is stable by the following gluing procedure.

Lemma 1.13. *Let $j_n\colon U_n \hookrightarrow M$ $(n \in \mathbb{N})$ be an increasing sequence of open embeddings of M with $\bigcup_n U_n = M$. We consider a sequence $\{F_n\}_n$ with $F_n \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{U_n})$ together with isomorphisms $u_{n+1,n}\colon F_n \xrightarrow{\sim} F_{n+1}|_{U_n}$. Then there exists $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ and isomorphisms $u_n\colon F|_{U_n} \simeq F_n$ such that $u_{n+1,n} = u_{n+1} \circ u_n^{-1}$ for all n. Moreover such a family $(F, \{u_n\}_n)$ is unique up to a (non-unique) isomorphism.*

Proof. (i) Denote by $v_n\colon j_{n!}(F_n) \to j_{n+1!}(F_{n+1})$ the morphisms obtained by adjunction. Then define $F \in \mathsf{D}(\mathbf{k}_M)$ as the homotopy colimit of this system, that is, F (which is defined up to isomorphism) is given by the distinguished triangle

$$\oplus_{n\in\mathbb{N}} j_{n!}(F_n) \xrightarrow{v := \oplus_{n\in\mathbb{N}}(\mathrm{id}_{j_{n!}(F_n)} - v_n)} \oplus_{n\in\mathbb{N}} j_{n!}(F_n) \to F \xrightarrow{+1} . \tag{1.16}$$

Then we have isomorphisms $u_n\colon F|_{U_n} \simeq F_n$ for all $n \in \mathbb{N}$, $u_{n+1,n} = u_{n+1} \circ u_n^{-1}$ and $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$.

(ii) Assume that we have another $G \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ and isomorphisms $w_n\colon G|_{U_n} \simeq F_n$. By adjunction they give $\varphi_n\colon j_{n!}(F_n) \to G$ and we let φ be the sum of the φ_n's. Since $u_{n+1,n} = w_{n+1} \circ w_n^{-1}$, we have $\varphi \circ v = 0$, where v is defined in (1.16). Hence φ factorizes through $\psi\colon F \to G$. Then $\psi|_{U_n} = w_n^{-1} \circ u_n$ is an isomorphism. The property of being an isomorphism being local, we obtain that ψ is an isomorphism. Q.E.D.

1.8 Quantized contact transformations ([13, §7.2])

Consider two manifolds M and N, two conic open subsets $U \subset T^*M$ and $V \subset T^*N$ and a homogeneous contact transformation χ:

$$T^*N \supset V \xrightarrow[\chi]{\sim} U \subset T^*M. \tag{1.17}$$

Denote by V^a the image of V by the antipodal map a_N on T^*N and by Λ the image of the graph of χ by $\mathrm{id}_U \times a_N$. Hence Λ is a conic Lagrangian submanifold of $U \times V^a$. A quantized contact transformation (a QCT, for short) above χ is a kernel $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times N})$ such that $\mathrm{SS}(K) \cap (U \times V^a) = \Lambda$ and satisfying some technical properties that we do not recall here so that the kernel K induces an equivalence of categories

$$K \circ \bullet : \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N; V) \xrightarrow{\sim} \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U). \tag{1.18}$$

Given χ and $q \in V$, $p = \chi(q) \in U$, there exists such a QCT after replacing U and V by sufficiently small neighborhoods of p and q.

1.9 The functor μhom ([13, §4.4, §7.2])

The functor of microlocalization along a submanifold has been introduced by Mikio Sato in the 70's and has been at the origin of what is now called "microlocal analysis". A variant of this functor, the bifunctor

$$\mu hom \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{T^*M}) \tag{1.19}$$

has been constructed in [13]. Since $\mathrm{Supp}(\mu hom(F, F')) \subset \mathrm{SS}(F) \cap \mathrm{SS}(F')$, (1.19) induces a bifunctor for any open subset U of T^*M:

$$\mu hom \colon \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U)^{\mathrm{op}} \times \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; U) \to \mathsf{D}^{\mathrm{b}}(\mathbf{k}_U).$$

Let us only recall the properties of this functor that we shall use. Consider a function $\psi \colon M \to \mathbb{R}$ defined in a neighborhood W of $x_0 \in M$ such that $d\psi(x_0) \neq 0$. Then, setting $S := \{x \in W\,;\, \psi(x) = \psi(x_0)\}$ and $p = d\psi(x_0)$, we have

$$\mathrm{R}\Gamma_{\{\psi(x)\geq\psi(x_0)\}}(F)_{x_0} \simeq \mu hom(\mathbf{k}_S, F)_p \quad \text{for any } F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M).$$

If χ is a contact transform as in (1.17) and if K is a QCT as in (1.18), then K induces a natural isomorphism for any $F, G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N; V)$

$$\chi_*(\mu hom(F, G)|_V) \xrightarrow{\sim} \mu hom(K \circ F, K \circ G)|_U. \tag{1.20}$$

1.10 Simple sheaves ([13, §7.5])

Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic Lagrangian submanifold and let $p \in \Lambda$. Simple sheaves along Λ at p are defined in [13, Def. 7.5.4].

When Λ is the conormal bundle to a submanifold $N \subset M$, that is, when the projection $\pi_M|_\Lambda \colon \Lambda \to M$ has constant rank, then an object $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ is simple along Λ at p if $F \simeq \mathbf{k}_N\,[d]$ in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_M; p)$ for some shift $d \in \mathbb{Z}$.

If $\mathrm{SS}(F)$ is contained in Λ on a neighborhood of Λ, Λ is connected and F is simple at some point of Λ, then F is simple at every point of Λ.

If $\Lambda_1 \subset T^*M_{12}$ and $\Lambda_2 \subset T^*M_{23}$ are locally closed conic Lagrangian submanifolds and if $K_i \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M_{i,i+1}})$ $(i = 1, 2)$ are simple along Λ_i, then $K_1 \circ K_2$ is simple along $\Lambda_1 \circ \Lambda_2$ under some conditions (see [13, Th. 7.5.11]). In particular, simple sheaves are stable by QCT.

Now, let M and N be two manifolds with the same dimension. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times N})$. Set

$$F^{-1} = v^{-1}\mathrm{R}\mathscr{H}om\,(F, \omega_M \boxtimes \mathbf{k}_N) \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{N\times M}), \tag{1.21}$$

where $v \colon N\times M \to M\times N$ is the swap. Let q_{ij} be the (i,j)-th projection from $N \times M \times N$. Then we have $F^{-1}\circ F = \mathrm{R}q_{13!}(q_{12}^{-1}F^{-1} \overset{\mathrm{L}}{\otimes} q_{23}^{-1}F)$. Let $\delta \colon N \to N \times N$ be the diagonal embedding. Then we have $\delta^{-1}(F^{-1}\circ F) \simeq \mathrm{R}q_{2!}(F \overset{\mathrm{L}}{\otimes} \mathrm{R}\mathscr{H}om\,(F, \omega_M \boxtimes \mathbf{k}_N))$. Hence $\delta^{-1}(F^{-1}\circ F) \simeq \mathrm{R}q_{2!}(F \overset{\mathrm{L}}{\otimes} \mathrm{R}\mathscr{H}om\,(F, q_2^{!}\mathbf{k}_N)) \to \mathrm{R}q_{2!}(q_2^{!}\mathbf{k}_N) \to \mathbf{k}_N$ gives a morphism

$$F^{-1}\circ F \to \mathbf{k}_{\Delta_N}.$$

Proposition 1.14 ([13, Proposition 7.1.8, Proposition 7.1.9, Theorem 7.2.1])**.** *Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times N})$, let $p_M \in \dot{T}^*M$ and let $p_N \in \dot{T}^*N$. Assume the following conditions:*

(i) $\mathrm{Supp}(F) \to N$ *is proper,*

(ii) F *is cohomologically constructible* (*see* [13, Def. 3.4.1]),

(iii) $\mathrm{SS}(F) \cap (\dot{T}^*M \times T^*_N N) = \emptyset$,

(iv) $\mathrm{SS}(F) \cap (T^*M \times \{p_N^a\}) = \{(p_M, p_N^a)\}$,

(v) $\mathrm{SS}(F)$ *is a Lagrangian submanifold of* $T^*(M \times N)$ *on a neighborhood of* (p_M, p_N^a),

(vi) F *is simple along* $\mathrm{SS}(F)$ *at* (p_M, p_N^a),

(vii) $\mathrm{SS}(F) \to T^*N$ *is a local isomorphism at* (p_M, p_N^a).

Then the morphism $F^{-1} \circ F \to \mathbf{k}_{\Delta_N}$ *is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{N\times N}; (p_N, p_N^a))$.

2 Deformation of the conormal to the diagonal

As usual, we denote by Δ_M or simply Δ the diagonal of $M \times M$. We denote by p_1 and p_2 the first and second projection from $T^*(M \times M)$ to T^*M and by p_2^a the composition of p_2 and the antipodal map on T^*M. We also set $n := \dim M$.

Consider a C^∞-function $f(x, y)$ defined on an open neighborhood $\Omega_0 \subset M \times M$ of the diagonal Δ_M. We assume that

(i) $f|_{\Delta_M} \equiv 0$,

(ii) $f(x, y) > 0$ for $(x, y) \in \Omega_0 \setminus \Delta_M$,

(iii) the partial Hessian $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(x, y)$ is positive definite for any $(x, y) \in \Delta_M$.

Such a pair $(\Omega_0,\ f)$ exists.

Proposition 2.1. *Assume that* $(\Omega_0,\ f)$ *satisfies the conditions* (i)–(iii) *above. Let* U *be a relatively compact open subset of* M. *Then there exist an* $\varepsilon > 0$ *and an open subset* Ω *of* $M \times M$ *satisfying the following conditions:*

(a) $\Delta_U \subset \Omega \subset \Omega_0 \cap (M \times U)$,

(b) $Z_\varepsilon := \{(x, y) \in \Omega\,;\, f(x, y) \leq \varepsilon\}$ *is proper over* U *by the map induced by the second projection,*

(c) *for any* $y \in U$ *and* $\varepsilon' \in]0, \varepsilon]$, *the open subset* $\{x \in M\,;\, (x, y) \in \Omega,\ f(x, y) < \varepsilon'\}$ *is homeomorphic to* $\mathbb{R}^n$,

(d) $d_x f(x,y) \neq 0$, $d_y f(x,y) \neq 0$ *for* $(x,y) \in \Omega \setminus \Delta_M$,

(e) *setting* $\Gamma_{Z_\varepsilon} = \{(x,y;\xi,\eta) \in T^*(\Omega)\,;\, f(x,y) = \varepsilon,\ (\xi,\eta) = \lambda df(x,y), \lambda < 0\}$, *the projection* $p_2^a \colon T^*(M \times U) \to T^*U$ *induces an isomorphism* $\Gamma_{Z_\varepsilon} \xrightarrow[p_2^a]{\sim} \dot{T}^*U$ *and the projection* $p_1 \colon T^*(M \times U) \to T^*M$ *induces an open embedding* $\Gamma_{Z_\varepsilon} \hookrightarrow \dot{T}^*M$.

Proof. Replacing Ω_0 with the open subset

$$\Delta_M \cup \{(x,y) \in \Omega_0\,;\, d_x f(x,y) \neq 0, d_y f(x,y) \neq 0\}\,,$$

we may assume from the beginning that Ω_0 satisfies (d).

Let $F \colon \Omega_0 \to T^*M$ be the map $(x,y) \mapsto d_y f(x,y)$. This map sends Δ_M to $T^*_M M$ and is a local isomorphism. Then there exists an open neighborhood $\Omega' \subset \Omega_0$ of Δ_M such that $F|_{\Omega'} \colon \Omega' \to T^*M$ is an open embedding. Hence by identifying Ω' as its image, we can reduce the proposition to the following lemma. Q.E.D.

Lemma 2.2. *Let* $p \colon E \to X$ *be a vector bundle of rank* n, $i \colon X \to E$ *the zero-section,* $SE = (E \setminus i(X))/\mathbb{R}_{>0}$ *the associated sphere bundle and* $q \colon E \setminus i(X) \to SE$ *the projection. Let* f *be a* C^∞*-function on a neighborhood* Ω *of the zero-section* $i(X)$ *of* E. *Assume the following conditions:*

(i) $f(z) = 0$ *for* $z \in i(X)$,

(ii) $f(z) > 0$ *for* $z \in \Omega \setminus i(X)$,

(iii) *for any* $x \in X$ *the Hessian of* $f|_{p^{-1}(x)}$ *at* $i(x)$ *is positive-definite.*

Then, for any relatively compact open subset U *of* X, *there exist* $\varepsilon > 0$ *and an open subset* $\Omega' \subset \Omega \cap p^{-1}(U)$ *containing* $i(U)$ *that satisfy the following conditions:*

(a) $\{z \in \Omega'\,;\, f(z) \leq \varepsilon\}$ *is proper over* U,

(b) $\{z \in \Omega'\,;\, 0 < f(z) < \varepsilon\} \to (SE \times_X U) \times]0,\varepsilon[$ *given by* $z \mapsto (q(z), f(z))$ *is an isomorphism,*

(c) *for any* $x \in U$ *and* $t \in]0,\varepsilon[$, *the set* $\{z \in \Omega' \cap p^{-1}(x)\,;\, f(z) < t\}$ *is homeomorphic to* $\mathbb{R}^n$.

Since the proof is elementary, we omit it.

Recall (1.21).

Theorem 2.3. *We keep the notations in Proposition 2.1 and set* $L = \mathbf{k}_{Z_\varepsilon} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times U})$*. Then* $\mathrm{SS}(L) \subset \Gamma_{Z_\varepsilon} \cup Z_\varepsilon$ *and* $L^{-1}\circ L \xrightarrow{\sim} \mathbf{k}_{\Delta_U}$*.*

Proof. Set $Z = Z_\varepsilon$. We have $\mathrm{SS}(L^{-1}\circ L) \subset T^*_{\Delta_U}(U\times U) \cup T^*_{U\times U}(U\times U)$. By Proposition 1.14, there exists a morphism $L^{-1}\circ L \to \mathbf{k}_{\Delta_U}$ which is an isomorphism in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{U\times U}; \dot{T}^*(U\times U))$. Hence if $N \to L^{-1}\circ L \to \mathbf{k}_{\Delta_U} \xrightarrow{+1}$ is a distinguished triangle, then $\mathrm{SS}(N) \subset T^*_{U\times U}(U\times U)$ and hence N has locally constant cohomologies. In particular $\mathrm{Supp}(N)$ is open and closed in $U\times U$. Let $\delta\colon U \to U\times U$ be the diagonal embedding. Then we have $\delta^{-1}(L^{-1}\circ L) \simeq \mathrm{R}q_{2!}(L \overset{\mathrm{L}}{\otimes} \mathrm{R}\mathscr{H}om(L, \mathbf{k}_{M\times U}) \overset{\mathrm{L}}{\otimes} q_2^!\mathbf{k}_U)$. Since $L \simeq \mathbf{k}_Z$ and $\mathrm{R}\mathscr{H}om(L, \mathbf{k}_{M\times U}) \simeq \mathbf{k}_{\mathrm{Int}(Z)}$, we have $\delta^{-1}(L^{-1}\circ L) \simeq \mathrm{R}q_{2!}(\mathbf{k}_{\mathrm{Int}(Z)} \overset{\mathrm{L}}{\otimes} q_2^!\mathbf{k}_U)$. Since the fibers of $\mathrm{Int}(Z) \to U$ are homeomorphic to $\mathbb{R}^n$, we have $\mathrm{R}q_{2!}(\mathbf{k}_{\mathrm{Int}(Z)} \overset{\mathrm{L}}{\otimes} q_2^!\mathbf{k}_U) \simeq \mathbf{k}_U$. Thus we obtain that $\delta^{-1}(L^{-1}\circ L) \simeq \mathbf{k}_U$, and hence $\delta^{-1}N \simeq 0$. Hence $\mathrm{Supp}(N) \cap \Delta_U = \emptyset$ and $\mathrm{Supp}(N) \subset \mathrm{Supp}(L^{-1}\circ L)$. Since we have

$$\mathrm{Supp}(L^{-1}\circ L) \subset \{(y, y') \in U\times U\,;\, (x,y),\,(x,y') \in Z \text{ for some } x \in M\}$$

and the fiber of $Z \to U$ is connected, y and y' belong to the same connected component of M as soon as $(y,y') \in \mathrm{Supp}(N)$. Since $\mathrm{Supp}(N)$ is open and closed in $U\times U$ and $\mathrm{Supp}(N)\cap\Delta_U = \emptyset$, we conclude that $\mathrm{Supp}(N) = \emptyset$. Q.E.D.

3 Quantization of homogeneous Hamiltonian isotopies

Let M be a real manifold of class C^∞ and I an open interval of $\mathbb{R}$ containing the origin. We consider a C^∞-map $\Phi\colon \dot{T}^*M\times I \to \dot{T}^*M$. Setting $\varphi_t = \Phi(\,\cdot\,, t)$ $(t\in I)$, we shall always assume

$$(3.1)\qquad \begin{cases} \varphi_t \text{ is a homogeneous symplectic isomorphism for each } t \in I, \\ \varphi_0 = \mathrm{id}_{\dot{T}^*M}\,. \end{cases}$$

Let us recall here some classical facts that we will explain with more details in Section A. Set

$$
\begin{aligned}
v_\Phi &:= \frac{\partial \Phi}{\partial t}\colon \dot{T}^*M \times I \to T\dot{T}^*M,\\
f &= \langle \alpha_M, v_\Phi \rangle\colon \dot{T}^*M \times I \to \mathbb{R},\ f_t = f(\cdot, t).
\end{aligned}
$$

Denote by H_g the Hamiltonian flow of a function $g\colon \dot{T}^*M \to \mathbb{R}$. Then

$$\frac{\partial \Phi}{\partial t} = H_{f_t}.$$

In other words, Φ is a homogeneous Hamiltonian isotopy.

In this situation, there exists a unique conic Lagrangian submanifold Λ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ closed in $\dot{T}^*(M \times M \times I)$ such that, setting

$$\Lambda_t = \Lambda \circ T^*_t I, \tag{3.2}$$

Λ_t is the graph of φ_t. (See Lemma A.2.)

The main result of this section is the existence and unicity of an object $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ whose microsupport is contained in Λ outside the zero-section and whose restriction at $t = 0$ is $\mathbf{k}_\Delta$. We shall call K the *quantization* of Φ on I or of $\{\varphi_t\}_{t\in I}$. We first prove that if such a K exists, then its support has some properness properties from which we deduce its unicity. Then we prove the existence assuming

$$\begin{cases}\text{there exists a compact subset } A \text{ of } M \text{ such that } \varphi_t \text{ is the}\\ \text{identity outside of } \dot{\pi}_M^{-1}(A) \text{ for all } t \in I,\ \dot{\pi}_M\colon \dot{T}^*M \to M\\ \text{denoting the projection.}\end{cases} \tag{3.3}$$

For this purpose we glue local constructions using the unicity. Then we prove the existence in general using an approximation of Φ by Hamiltonian isotopies satisfying (3.3).

3.1 Unicity and support of the quantization

We introduce the notations

$$
\begin{aligned}
&I_t = [0,t] \text{ or } [t,0] \text{ according to the sign of } t \in I,\\
&B := \{(x,y,t) \in M \times M \times I\,;\, (\{x\} \times \{y\} \times I_t) \cap \dot{\pi}_{M\times M\times I}(\Lambda) \neq \emptyset\}\,.
\end{aligned} \tag{3.4}
$$

Lemma 3.1. *Both projections $B \rightrightarrows M \times I$ are proper.*

Proof. (i) Let us show that the second projection $q\colon B \to M \times I$ is proper. We see easily that $q^{-1}(y,t) = \dot\pi_M(\Phi(\dot\pi_M^{-1}(y) \times I_t)) \times \{y\} \times \{t\}$. We choose a compact set $D \subset M$ and $t \in I$. Then $q^{-1}(D \times I_t)$ is contained in $\dot\pi_M(\Phi(\dot\pi_M^{-1}(D) \times I_t)) \times D \times I_t$ which is compact.
(ii) The first projection is treated similarly by reversing the roles of x and y and replacing Φ by $\Phi^{-1} = \{\varphi_t^{-1}\}_{t \in I}$. Q.E.D.

Recall that for $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times N})$, the object F^{-1} is defined in (1.21). For an object $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ and $t_0 \in I$, we set

$$K_{t_0} = K|_{t=t_0} \simeq K \circ \mathbf{k}_{t=t_0} \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M}).$$

We also set (keeping the same notation for v as in (1.21)):

$$K^{-1} = (v \times \mathrm{id}_I)^{-1} \mathrm{R}\mathscr{H}om\,(K, \omega_M \boxtimes \mathbf{k}_M \boxtimes \mathbf{k}_I).$$

Then assuming

$$\mathrm{SS}(K) \cap T^*_{M\times M}(M \times M) \times T^*I \subset T^*_{M\times M\times I}(M \times M \times I),$$

we have $(K^{-1})_t \simeq (K_t)^{-1}$ for any $t \in I$. (See the proof of (ii) in the proposition below.)

Proposition 3.2. *We assume that Φ satisfies hypothesis* (3.1) *and that $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ satisfies the following conditions.*

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$,

(b) $K_0 \simeq \mathbf{k}_\Delta$.

Then we have:

(i) $\mathrm{Supp}(K) \subset B$ (*see* (3.4)) *and both projections* $\mathrm{Supp}(K) \rightrightarrows M \times I$ *are proper,*

(ii) $K_t \circ K_t^{-1} \simeq K_t^{-1} \circ K_t \simeq \mathbf{k}_\Delta$ *for all* $t \in I$,

(iii) *such a K satisfying the conditions* (a)–(b) *is unique up to a unique isomorphism,*

(iv) *if there exists an open set* $W \subset M$ *such that* $\varphi_t|_{\dot\pi_M^{-1}(W)} = \mathrm{id}$ *for all* $t \in I$, *then* $K|_{(W\times M \cup M\times W)\times I} \simeq \mathbf{k}_{\Delta\times I}|_{(W\times M\cup M\times W)\times I}$.

Proof. (i) Let us prove (i). Since Λ is closed and conic, $\dot\pi_{M\times M\times I}(\Lambda)$ is closed. So if $(x,y,t)\notin B$ we may find open connected neighborhoods U of x, V of y and J of I_t such that $\dot\pi^{-1}_{M\times M\times I}(U\times V\times J)$ does not meet Λ. By condition (a) this implies that $\mathrm{SS}(K|_{U\times V\times J})$ is contained in the zero-section. Hence K is locally constant on $U\times V\times J$. Now $0\in J$ and $U\times V$ does not meet Δ_M since $\dot\pi_{M\times M\times I}(\Lambda)$ contains $\Delta_M\times\{0\}$. Hence $K|_{U\times V\times\{0\}}=0$ and we deduce $K|_{U\times V\times J}=0$. In particular $(x,y,t)\notin\mathrm{Supp}(K)$ and this proves $\mathrm{Supp}(K)\subset B$. To conclude, we apply Lemma 3.1.

(ii) Let us prove (ii). We set $F = K\circ|_I K^{-1}$ (Notation (1.13)). Hence (ii) is implied by $F\simeq\mathbf{k}_{\Delta\times I}$. Let v be the involution of $T^*M\times T^*M\times T^*I$ given by $v(x,\xi,x',\xi',t,\tau)=(x',-\xi',x,-\xi,t,-\tau)$. Then we have

$$\mathrm{SS}(K^{-1})\cap\dot T^*(M\times M\times I)\subset v(\Lambda). \tag{3.5}$$

Hence by (1.15), $\mathrm{SS}(F)$ satisfies:

$$\begin{aligned}\mathrm{SS}(F) &\subset T^*_{\Delta_M\times I}(M\times M\times I)\cup T^*_{M\times M\times I}(M\times M\times I)\\ &\subset T^*(M\times M)\times T^*_I I.\end{aligned}$$

By Corollary 1.6, F is constant on the fibers of $M\times M\times I\to M\times M$. Denote by $i_0\colon M\times M\to M\times M\times I$ the inclusion associated to $\{t=0\}\subset I$. It is thus enough to prove the isomorphism $i_0^{-1}F\simeq\mathbf{k}_\Delta$. We have

$$\begin{aligned} i_0^!\mathrm{R}\mathscr{H}om(K,\mathbf{k}_{M\times M\times I}) &\simeq \mathrm{R}\mathscr{H}om(i_0^{-1}K, i_0^!\mathbf{k}_{M\times M\times I})\\ &\simeq \mathrm{R}\mathscr{H}om(K_0,\mathbf{k}_{M\times M})\overset{\mathrm{L}}{\otimes} i_0^!\mathbf{k}_{M\times M\times I}.\end{aligned}$$

On the other hand, the condition on $\mathrm{SS}(K)$ implies

$$i_0^!\mathrm{R}\mathscr{H}om(K,\mathbf{k}_{M\times M\times I})\simeq i_0^{-1}\mathrm{R}\mathscr{H}om(K,\mathbf{k}_{M\times M\times I})\overset{\mathrm{L}}{\otimes} i_0^!\mathbf{k}_{M\times M\times I}.$$

Therefore $i_0^{-1}\mathrm{R}\mathscr{H}om(K,\mathbf{k}_{M\times M\times I})\simeq\mathrm{R}\mathscr{H}om(K_0,\mathbf{k}_{M\times M})$ which gives the isomorphism $i_0^{-1}K^{-1}\simeq K_0^{-1}$. Thus we obtain $i_0^{-1}F\simeq K_0\circ K_0^{-1}\simeq\mathbf{k}_\Delta$ as required.

(iii) is a particular case of the more precise Lemma 3.3 below.

(iv) We set $\widetilde W = (W\times M\cup M\times W)\times I$. Then $B\cap\widetilde W=\Delta_W\times I$. Hence (i) implies that $\mathrm{Supp}(K)\cap\widetilde W\subset\Delta_W\times I$. Then (b) implies (iv). Q.E.D.

Lemma 3.3. *Let* $\Phi_i\colon \dot{T}^*M \times I \to \dot{T}^*M$ $(i = 1, 2)$ *be two maps satisfying* (3.1) *and define* $\Lambda_i \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ *as in Lemma A.2. Assume that there exist* $K_i \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ $(i = 1, 2)$ *satisfying conditions* (a)–(b) *of Proposition 3.2. Also assume that there exists an open set* $U \subset M$ *such that*

$$\Phi_1|_{\dot{\pi}_M^{-1}(U)\times I} = \Phi_2|_{\dot{\pi}_M^{-1}(U)\times I}. \tag{3.6}$$

Then there exists a unique isomorphism $\psi\colon K_1|_{M\times U\times I} \xrightarrow{\sim} K_2|_{M\times U\times I}$ *such that*

$$\begin{tikzcd} i_0^{-1}K_1 \arrow[rr, "i_0^{-1}\psi"] \arrow[dr, "\sim"'] & & i_0^{-1}K_2 \arrow[dl, "\sim"] \\ & \mathbf{k}_{\Delta_U} & \end{tikzcd}$$

commutes, where $i_0\colon M \times U \to M \times U \times I$ *is the inclusion by* $0 \in I$.

Proof. We define $\Phi_2^{-1}\colon \dot{T}^*M \times I \to \dot{T}^*M$ by $\varphi_{2,t}^{-1} = (\varphi_{2,t})^{-1}$ for all $t \in I$. Then, similarly to (3.5), we have $\mathrm{SS}(K_2^{-1}) \subset v(\Lambda_2)$ outside the zero-section. We also define $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ by $\varphi_t = \varphi_{2,t}^{-1} \circ \varphi_{1,t}$. Its associated Lagrangian submanifold is $\Lambda = v(\Lambda_2) \circ|_I \Lambda_1$ (see (1.15)). By (3.6) we have $\varphi_t(x, \xi) = (x, \xi)$ for all $t \in I$ and $(x, \xi) \in \dot{\pi}_M^{-1}(U)$. Hence

$$\Lambda \cap \dot{T}^*(M \times U \times I) = \dot{T}^*_{\Delta_U}(M \times U) \times T^*_I I.$$

We set $L = K_2^{-1} \circ|_I K_1$. By (1.15), we have the inclusion $\mathrm{SS}(L) \subset \Lambda$ outside the zero-section. It follows that $\mathrm{SS}(L)$ is contained in $T^*(M \times U) \times T^*_I I$ over $M \times U \times I$. Since $L_0 \simeq \mathbf{k}_{\Delta_M}$, we deduce from Corollary 1.6 that $L|_{M\times U\times I} \simeq \mathbf{k}_{(\Delta_M \cap M\times U)\times I}$. Then

$$K_1|_{M\times U\times I} \simeq (K_2 \circ|_I L)|_{M\times U\times I} \simeq K_2 \circ|_I (L|_{M\times U\times I}) \simeq K_2|_{M\times U\times I}$$

as claimed.

The uniqueness of ψ follows from the uniqueness of the isomorphism $L|_{M\times U\times I} \simeq \mathbf{k}_{\Delta_U\times I}$. Q.E.D.

3.2 Existence of the quantization – "compact" case

Lemma 3.5 below is the main step in the proof of Theorem 3.7. We prove the existence of a quantization of a homogeneous Hamiltonian isotopy $\Phi\colon \dot{T}^*M \times$

$I \to \dot{T}^*M$ satisfying hypothesis (3.1) and (3.3) (that is φ_t is the identity map outside $\dot{\pi}_M^{-1}(A)$ for each $t \in I$, where $A \subset M$ is compact). In the course of the proof we shall need an elementary lemma that we state without proof.

Lemma 3.4. *Let N be a manifold, $V_0 \subset N$ an open subset with a smooth boundary, $C \subset N$ a compact subset and I an open interval of $\mathbb{R}$ containing 0. Let $\Lambda \subset \dot{T}^*(N \times I)$ be a closed conic Lagrangian submanifold and set $\Lambda_t = \Lambda \circ T_t^* I$ for $t \in I$. We assume*

(a) $\Lambda_0 = \mathrm{SS}(\mathbf{k}_{\overline{V_0}}) \cap \dot{T}^*N$,

(b) $\Lambda \cap \dot{T}^*((N \setminus C) \times I) = (\Lambda_0 \cap \dot{T}^*(N \setminus C)) \times T_I^* I$,

(c) $\Lambda \subset \dot{T}^*N \times T^*I$ *and* $\Lambda \to \dot{T}^*N \times I$ *is a closed embedding.*

Then there exist $\varepsilon > 0$ with $\pm\varepsilon \in I$ and an open subset $V \subset N \times]-\varepsilon, \varepsilon[$ with a smooth boundary such that

(i) $V_0 = V \cap (N \times \{0\})$,

(ii) $\Lambda = \mathrm{SS}(\mathbf{k}_{\overline{V}}) \cap \dot{T}^*(N \times]-\varepsilon, \varepsilon[)$,

(iii) $\Lambda_t = \mathrm{SS}(\mathbf{k}_{\overline{V \cap (N \times \{t\})}}) \cap \dot{T}^*N$ *for any* $t \in]-\varepsilon, \varepsilon[$.

Lemma 3.5. *Assume that Φ satisfies hypotheses* (3.1) *and* (3.3). *Then there exists $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M \times M \times I})$ satisfying conditions* (a)–(b) *of Proposition 3.2.*

Proof. (A) Local existence. We first prove that there exists $\varepsilon > 0$ such that there exists a quantization $\widetilde{K} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M \times M \times]-\varepsilon,\varepsilon[})$ of Φ on $]-\varepsilon, \varepsilon[$.

We use the results and notations of Proposition 2.1 and Theorem 2.3. We choose a relatively compact open subset U such that $A \subset U \subset M$ where A is given in hypothesis (3.3). We choose f and ε_1 as in Proposition 2.1 (in which ε_1 was denoted by ε). Then $L := \mathbf{k}_{Z_{\varepsilon_1}} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M \times U})$ satisfies $\mathrm{SS}(L) = \Gamma_{Z_{\varepsilon_1}} \cup Z_{\varepsilon_1}$, and $L^{-1} \circ L \simeq \mathbf{k}_{\Delta_U}$. We define for $t \in I$

$$\widetilde{\Lambda} := \Gamma_{Z_{\varepsilon_1}} \circ \Lambda \subset \dot{T}^*M \times \dot{T}^*U \times T^*I,$$

$$\widetilde{\Lambda}_t := \Gamma_{Z_{\varepsilon_1}} \circ \Lambda_t \subset \dot{T}^*M \times \dot{T}^*U.$$

We remark that $\widetilde{\Lambda}_t = \widetilde{\Lambda} \circ T_t^* I$ and $\widetilde{\Lambda}_0 = \mathrm{SS}(\mathbf{k}_{Z_{\varepsilon_1}}) \cap \dot{T}^*(U \times I)$. We apply Lemma 3.4 with $N = M \times U$, $C = A \times A$, $V_0 = \mathrm{Int} Z_{\varepsilon_1}$. We obtain $\varepsilon > 0$ and an open subset $V \subset M \times U \times]-\varepsilon, \varepsilon[$ such that $\widetilde{L} := \mathbf{k}_{\overline{V}} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M \times U \times]-\varepsilon,\varepsilon[})$ satisfies:

(a) $\mathrm{SS}(\widetilde{L}) \subset (\widetilde{\Lambda}\times_I]-\varepsilon,\varepsilon[) \cup T^*_{M\times U\times]-\varepsilon,\varepsilon[}(M\times U\times]-\varepsilon,\varepsilon[)$,

(b) $\widetilde{L}|_{M\times U\times\{0\}} \simeq \mathbf{k}_{Z_{\varepsilon_1}}$,

(c) the projection $M\times U\times]-\varepsilon,\varepsilon[\to U\times]-\varepsilon,\varepsilon[$ is proper on $\mathrm{Supp}(\widetilde{L})$.

Now we set

$$K = L^{-1}\circ|_I \widetilde{L} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{U\times U\times]-\varepsilon,\varepsilon[}).$$

Then K satisfies properties (a)–(b) of Proposition 3.2 when replacing M and I with U and $]-\varepsilon,\varepsilon[$. We deduce in particular

$$K|_{((U\times U)\setminus(A\times A))\times]-\varepsilon,\varepsilon[} \simeq (\mathbf{k}_{\Delta_M\times]-\varepsilon,\varepsilon[})|_{((U\times U)\setminus(A\times A))\times]-\varepsilon,\varepsilon[}.$$

Applying Lemma 1.11, K extends to $\tilde{K}\in\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times]-\varepsilon,\varepsilon[})$ with

$$\tilde{K}|_{((M\times M)\setminus(A\times A))\times]-\varepsilon,\varepsilon[} \simeq (\mathbf{k}_{\Delta_M\times]-\varepsilon,\varepsilon[})|_{((M\times M)\setminus(A\times A))\times]-\varepsilon,\varepsilon[}$$

and $\tilde{K}\in\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times]-\varepsilon,\varepsilon[})$ is a quantization of Φ on $]-\varepsilon,\varepsilon[$.

(B) Gluing (a). Assume $K^{t_0,t_1}\in\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times]t_0,t_1[})$ is a quantization of the isotopy $\{\varphi_t\}_{t\in]t_0,t_1[}$ for an open interval $J=]t_0,t_1[\subset I$ containing the origin.

Assume that $J\neq I$. We shall show that there exist an open interval $J'\subset I$ and a quantization of the isotopy $\{\varphi_t\}_{t\in J'}$ such that $J\subset J'$ and $J'\neq J$.

For an interval $I'\subset]t_0,t_1[$, we write $K^{t_0,t_1}|_{I'}$ for $K^{t_0,t_1}|_{M\times M\times I'}$.

Assume that $t_1\in I$. By applying the result of (A) to the isotopy $\{\varphi_t\circ\varphi_{t_1}^{-1}\}_{t\in I}$, there exist $t_0<t_3<t_1<t_4$ with $t_4\in I$ and a quantization $L^{t_3,t_4}\in\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times]t_3,t_4[})$ of the isotopy $\{\varphi_t\circ\varphi_{t_1}^{-1}\}_{t\in]t_3,t_4[}$. Choose t_2 with $t_3<t_2<t_1$ and set

$$\begin{aligned} F &= (K^{t_0,t_1}|_{]t_3,t_1[})\circ(K^{t_0,t_1}_{t_2})^{-1},\\ F' &= (L^{t_3,t_4}|_{]t_3,t_1[})\circ(L^{t_3,t_4}_{t_2})^{-1}. \end{aligned}$$

Then both F and F' are a quantization of the isotopy $\{\varphi_t\circ\varphi_{t_2}^{-1}\}_{t\in]t_3,t_1[}$. By unicity of the quantization (Proposition 3.2), F and F' are isomorphic and hence we have an isomorphism

$$K^{t_3,t_4}|_{]t_3,t_1[}\simeq K^{t_0,t_1}|_{]t_3,t_1[} \text{ in } \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times]t_3,t_1[}),$$

where $K^{t_3,t_4} = L^{t_3,t_4} \circ (L^{t_3,t_4}_{t_2})^{-1} \circ K^{t_0,t_1}_{t_2} \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times]t_3,t_4[})$. By Lemma 1.11 there exists $K^{t_0,t_4} \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times]t_0,t_4[})$ such that $K^{t_0,t_4}|_{]t_0,t_1[} \simeq K^{t_0,t_1}$ and $K^{t_0,t_4}|_{]t_3,t_4[} \simeq K^{t_3,t_4}$. Then K^{t_0,t_4} is a quantization of the isotopy $\{\varphi_t\}_{t\in]t_0,t_4[}$. Similarly, if $t_0 \in I$, then there exists $t_{-1} \in I$ with $t_{-1} < t_0$ and a quantization K^{t_{-1},t_1} on $]t_{-1}, t_1[$.

(C) Gluing (b). Consider an increasing sequence of open intervals $I_n \subset I$ and assume we have constructed quantizations K_n of $\{\varphi_t\}_{t\in I_n}$. By unicity in Proposition 3.2 we have $K_{n+1}|_{M\times M\times I_n} \simeq K_n$. Set $J = \bigcup_n I_n$. By Lemma 1.13 there exists $K_J \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times J})$ such that $K_J|_{M\times M\times I_n} \simeq K_n$. Then K_J is a quantization of $\{\varphi_t\}_{t\in J}$.

(D) Consider the set of pairs (J, K_J) where J is an open interval contained in I and containing 0 and K_J is a quantization of $\{\varphi_t\}_{t\in J}$. This set, ordered by inclusion, is inductively ordered by (C). Let (J, K_J) be a maximal element. It follows from (B) that $J = I$. Q.E.D.

3.3 Existence of the quantization – general case

In this section we remove hypothesis (3.3) in Lemma 3.5. We consider $\Phi\colon \dot{T}^*M\times I \to \dot{T}^*M$ which only satisfies (3.1) and we consider $f\colon \dot{T}^*M\times I \to \mathbb{R}$ and $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ as above. We will define approximations of Φ by Hamiltonian isotopies satisfying (3.3) such that their quantizations "stabilize" over compact sets which allows us to define the quantization of Φ.

We consider a C^∞-function $g\colon M \to \mathbb{R}$ with compact support. The function

$$f_g\colon \dot{T}^*M \times I \to \mathbb{R}, \qquad (x,\xi,t) \mapsto g(x)f(x,\xi,t)$$

is homogeneous of degree 1 and has support in $\dot{\pi}_M^{-1}(A) \times I$ with $A = \mathrm{supp}(g)$ compact. So its Hamiltonian flow is well-defined and satisfies (3.1) and (3.3). We denote it by

$$\Phi_g\colon \dot{T}^*M \times I \to \dot{T}^*M$$

and we let $\Lambda_g \subset \dot{T}^*M\times\dot{T}^*M\times T^*I$ be the Lagrangian submanifold associated to Φ_g in Lemma A.2. By Lemma 3.5 there exists a unique $K_g \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ such that

$$\mathrm{SS}(K_g) \subset \Lambda_g \cup T^*_{M\times M\times I}(M\times M\times I) \quad \text{and} \quad K_g|_{M\times M\times\{0\}} \simeq \mathbf{k}_{\Delta_M}.$$

Lemma 3.6. *Let $U \subset M$ be a relatively compact open subset and $J \subset I$ a relatively compact open subinterval. Then there exists a C^∞-function $g\colon M \to \mathbb{R}$ with compact support such that*

$$\Phi_g|_{\dot\pi_M(U)\times J} = \Phi|_{\dot\pi_M(U)\times J}. \tag{3.7}$$

Proof. We assume without loss of generality that $0 \in J$. Since Φ is homogeneous, the set $C := \dot\pi_M(\Phi(\dot\pi_M^{-1}(\overline{U}) \times \overline{J}))$ is a compact subset of M. We choose a C^∞-function $g\colon M \to \mathbb{R}$ with compact support such that g is equal to the constant function 1 on some neighborhood of C.

Then for any $p \in \dot\pi_M^{-1}(U)$ the functions f and f_g coincide on a neighborhood of $\gamma_{p,J} := \{(\varphi_t(p), t);\ t \in J\} \subset \dot T^*M \times I$ which is the trajectory of p by the flow Φ. Hence their Hamiltonian vector fields coincide on $\gamma_{p,J}$ and their flows also. Q.E.D.

Theorem 3.7. *We consider $\Phi\colon \dot T^*M \times I \to \dot T^*M$ and we assume that it satisfies hypothesis* (3.1). *Then there exists $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ satisfying conditions* (a)–(b) *of Proposition 3.2.*

Proof. We consider an increasing sequence of relatively compact open subsets $U_n \subset M$, $n \in \mathbb{N}$, and an increasing sequence of relatively compact open subintervals $J_n \subset I$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} U_n = M$ and $\bigcup_{n\in\mathbb{N}} J_n = I$. By Lemma 3.6 we can choose C^∞-functions $g_n\colon M \to \mathbb{R}$ with compact supports such that Φ_{g_n} and Φ coincide on $\dot\pi_M(U_n) \times J_n$. We let $K_n \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ be the quantization of Φ_{g_n}. In particular

$$\begin{aligned} &\mathrm{SS}(K_n) \cap \dot T^*(M \times U_n \times J_n) \subset \Lambda \cap \dot T^*(M \times U_n \times J_n), \\ &K_n|_{M\times M\times\{0\}} \simeq \mathbf{k}_{\Delta_M}. \end{aligned} \tag{3.8}$$

By Lemma 3.3 we have isomorphisms $K_{n+1}|_{M\times U_n\times J_n} \simeq K_n|_{M\times U_n\times J_n}$. Then Lemma 1.13 implies that there exists $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ with isomorphisms $K|_{M\times U_n\times J_n} \simeq K_n|_{M\times U_n\times J_n}$ for any $n \in \mathbb{N}$. Then K satisfies (a)–(b) of Proposition 3.2 by (3.8). Q.E.D.

Remark 3.8. If Φ also satisfies (3.3) and J is a relatively compact subinterval of I, then the restriction $K|_{M\times M\times J}$ belongs to $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times J})$. See Example 3.11.

Remark 3.9. Theorem 3.7 extends immediately when replacing the open interval I with a smooth contractible manifold U with a marked point u_0.

Indeed, consider a homogeneous Hamiltonian isotopy $\Phi\colon \dot T^*M \times U \to \dot T^*M$ with $\varphi_{u_0} = \mathrm{id}_{\dot T^*M}$. We can construct a Lagrangian submanifold Λ of $\dot T^*(M \times M \times U)$ as in Lemma A.2. Set $\delta = \mathrm{id}_{M\times M} \times \delta_U$ where $\delta_U\colon U \to U \times U$ is the diagonal embedding. One easily sees that $\delta_\pi \delta_d^{-1}(\Lambda)$ is the graph of a homogeneous symplectic diffeomorphism $\widetilde\Phi$ of $\dot T^*(M \times U)$.

Now let $h\colon U \times I \to U$ be a retraction to u_0 such that $h(U \times \{0\}) = \{u_0\}$ and $h|_{U\times\{1\}} = \mathrm{id}_U$. We set $\Phi_t = \Phi \circ (\mathrm{id}_{\dot T^*M} \times h_t)$ and we apply the above procedure to each Φ_t. Then $\{\widetilde\Phi_t\}_{t\in I}$ is a homogeneous Hamiltonian isotopy of $\dot T^*(M \times U)$ with $\widetilde\Phi_0 = \mathrm{id}$ and Theorem 3.7 associates with it a kernel $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times U\times M\times U\times I})$. One checks that K is supported on $M\times M\times\Delta_U\times I$ and that $K|_{M\times M\times\Delta_U\times\{1\}}$ is the desired kernel.

Example 3.10. Let $M = \mathbb{R}^n$ and denote by $(x;\xi)$ the homogeneous symplectic coordinates on $T^*\mathbb{R}^n$. Consider the isotopy $\varphi_t(x;\xi) = (x - t\frac{\xi}{|\xi|};\xi)$, $t \in I = \mathbb{R}$. Then

$$\begin{aligned}&\Lambda_t = \{(x,y,\xi,\eta); |x-y| = |t|,\ \xi = -\eta = s(x-y),\ st < 0\} \quad \text{for } t \neq 0,\\ &\Lambda_0 = \dot T^*_\Delta(M \times M).\end{aligned}$$

The isomorphisms

$$\begin{aligned}\mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\Delta\times\{t=0\}}, \mathbf{k}_{M\times M\times\mathbb{R}}) &\simeq \mathbf{k}_{\Delta\times\{t=0\}}[-n-1]\\ \mathrm{R}\mathscr{H}om\,(\mathbf{k}_{\{|x-y|\le -t\}}, \mathbf{k}_{M\times M\times\mathbb{R}}) &\simeq \mathbf{k}_{\{|x-y|<-t\}}\end{aligned}$$

together with the morphism $\mathbf{k}_{\{|x-y|\le -t\}} \to \mathbf{k}_{\Delta\times\{t=0\}}$ induce the morphism $\mathbf{k}_{\Delta\times\{t=0\}}[-n-1] \to \mathbf{k}_{\{|x-y|<-t\}}$. Hence we obtain

$$\mathbf{k}_{\{|x-y|\le t\}} \to \mathbf{k}_{\Delta\times\{t=0\}} \to \mathbf{k}_{\{|x-y|<-t\}}[n+1].$$

Let ψ be the composition. Then there exists a distinguished triangle in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$:

$$\mathbf{k}_{\{|x-y|<-t\}}[n] \to K \to \mathbf{k}_{\{|x-y|\le t\}} \xrightarrow[\psi]{+1}$$

We can verify that K satisfies the properties (a)–(b) of Proposition 3.2. From this distinguished triangle, we deduce the isomorphisms in $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M})$: $K_t \simeq \mathbf{k}_{\{|x-y|\le t\}}$ for $t \ge 0$ and $K_t \simeq \mathbf{k}_{\{|x-y|<-t\}}[n]$ for $t < 0$.

Example 3.11. We shall give an example where the quantization $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ of a Hamiltonian isotopy does not belong to $\mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$. Let us take the n-dimensional unit sphere $M = \mathbb{S}^n$ ($n \geq 2$) endowed with the canonical Riemannian metric. The metric defines an isomorphism $T^*M \simeq TM$ and the length function $f\colon \dot{T}^*M \to \mathbb{R}$. Then f is a positive-valued function on $\dot{T}^*M$ homogeneous of degree 1. Set $I = \mathbb{R}$ and let $\Phi = \{\varphi_t\}_{t\in I}$ be the Hamiltonian isotopy associated with f. Then for $p \in \dot{T}^*M$, $\{\pi_M(\varphi_t(p))\}_{t\in I}$ is a geodesic. For $x, y \in M$, $\mathrm{dist}(x, y)$ denotes the distance between x and y. Let $a\colon M \to M$ be the antipodal map. Then we have $\mathrm{dist}(x, y)+\mathrm{dist}(x, y^a) = \pi$. For any integer ℓ we set

$$C_\ell = \begin{cases} \big\{(x,y,t) \in M \times M \times \mathbb{R}\,;\, t \geq \ell\pi \text{ and } \mathrm{dist}(x, a^\ell(y)) \leq t - \ell\pi\big\} & \text{if } \ell \geq 0, \\[2ex] \left\{(x,y,t) \in M \times M \times \mathbb{R}\,;\, \begin{array}{l} t < (\ell+1)\pi \text{ and} \\ \mathrm{dist}(x, a^{\ell+1}(y)) < -t + (\ell+1)\pi \end{array}\right\} & \text{if } \ell < 0. \end{cases}$$

Let K be the quantization of Φ. Then we have

$$H^k(K) \simeq \begin{cases} \mathbf{k}_{C_\ell} & \text{if } k = (n-1)\ell \text{ for some } \ell \in \mathbb{Z}_{\geq 0}, \\ \mathbf{k}_{C_\ell} & \text{if } k = (n-1)\ell - 1 \text{ for some } \ell \in \mathbb{Z}_{<0}, \\ 0 & \text{otherwise.} \end{cases}$$

3.4 Deformation of the microsupport

We consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ satisfying hypothesis (3.1) as in Theorem 3.7, we denote as usual by Λ the Lagrangian submanifold associated to its graph and define Λ_t as in (3.2). Let $S_0 \subset \dot{T}^*M$ be a closed conic subset. We set $S = \Lambda \circ S_0$ and $S_t = \Lambda_t \circ S_0$, closed conic subsets of $\dot{T}^*M \times T^*I$ and $\dot{T}^*M$ respectively. We let $i_t\colon M \to M \times I$ be the natural inclusion for $t \in I$. Consider the functor

$$(3.9) \qquad i_t^{-1}\colon \mathsf{D}^{\mathrm{lb}}_{S\cup T^*_{M\times I}(M\times I)}(\mathbf{k}_{M\times I}) \to \mathsf{D}^{\mathrm{lb}}_{S_t\cup T^*_M M}(\mathbf{k}_M).$$

Proposition 3.12. *For any $t \in I$ the functor* (3.9) *is an equivalence of categories. In particular for any $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ such that $\mathrm{SS}(F) \subset S_t \cup T^*_M M$ there exists a unique $G \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times I})$ such that $i_t^{-1}(G) \simeq F$ and $\mathrm{SS}(G) \subset S \cup T^*_{M\times I}(M \times I)$. If Φ also satisfies* (3.3) *and we replace I by a relatively compact subinterval, then* (3.9) *induces an equivalence between bounded derived categories.*

Proof. Replacing Φ by $\Phi \circ \varphi_t^{-1}$ we may as well assume that $t = 0$. Let $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ be the quantization of Φ. Then we obtain the commutative diagram:

$$\begin{tikzcd}
\mathsf{D}^{\mathrm{lb}}_{(S_0\times T^*_I I)\cup T^*_{M\times I}(M\times I)}(\mathbf{k}_{M\times I}) \arrow[r, "K\circ|_I\cdot", "\sim"'] \arrow[d, "r_0"'] & \mathsf{D}^{\mathrm{lb}}_{S\cup T^*_{M\times I}(M\times I)}(\mathbf{k}_{M\times I}) \arrow[d, "i_0^{-1}"] \\
\mathsf{D}^{\mathrm{lb}}_{S_0\cup T^*_M M}(\mathbf{k}_M) \arrow[r, equal] & \mathsf{D}^{\mathrm{lb}}_{S_0\cup T^*_M M}(\mathbf{k}_M),
\end{tikzcd}$$

where r_0 is induced by i_0^{-1}. It is known that r_0 is an equivalence of categories, with inverse given by q_1^{-1}, where $q_1\colon M\times I \to M$ is the projection. Hence so is the morphism i_0^{-1} defined by (3.9).

The last assertion follows from Remark 3.8. Q.E.D.

Now we consider a closed conic Lagrangian submanifold $S_0 \subset \dot{T}^*M$ and a deformation of S_0 indexed by I, $\Psi\colon S_0\times I \to \dot{T}^*M$ as in Definition A.3. We let $S \subset \dot{T}^*M \times T^*I$ be the corresponding Lagrangian submanifold defined in Lemma A.4. Then Propositions A.5 and 3.12 imply the following result.

Corollary 3.13. *We consider a deformation $\Psi\colon S_0\times I \to \dot{T}^*M$ as in Definition A.3 and we assume that it satisfies* (A.7). *Then for any $t \in I$ the functor* (3.9) *is an equivalence of categories. Moreover if we replace I by a relatively compact subinterval, it induces an equivalence between the bounded derived categories.*

4 Applications to non-displaceability

We denote by $\Phi = \{\varphi_t\}_{t\in I}\colon \dot{T}^*M\times I \to \dot{T}^*M$ a homogeneous Hamiltonian isotopy as in Theorem 3.7. Hence, Φ satisfies hypothesis (3.1).

Let $F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$. We assume

$$F_0 \text{ has a compact support.} \tag{4.1}$$

We let $\Lambda \subset \dot{T}^*(M\times M\times I)$ be the conic Lagrangian submanifold associated to Φ in Lemma A.2 and we let $K \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$ be the quantization of Φ on I constructed in Theorem 3.7. We set:

$$\begin{aligned} &F = K\circ F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{M\times I}),\\ &F_{t_0} = F|_{\{t=t_0\}} \simeq K_{t_0}\circ F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M) \quad \text{for } t_0\in I. \end{aligned} \tag{4.2}$$

Then

$$
(4.3)\quad \begin{cases} \mathrm{SS}(F) \subset (\Lambda \circ \mathrm{SS}(F_0)) \cup T^*_{M\times I}(M\times I), \\ \mathrm{SS}(F) \cap T^*_M M \times T^* I \subset T^*_{M\times I}(M\times I), \\ \text{the projection } \mathrm{Supp}(F) \to I \text{ is proper.} \end{cases}
$$

The first assertion follows from (1.12), and the second assertion follows from the first. The third one follows from Proposition 3.2 (i) and (4.1). In particular we have

$$
(4.4)\quad \begin{cases} F_t \text{ has a compact support in } M, \\ \mathrm{SS}(F_t) \cap \dot{T}^* M = \varphi_t(\mathrm{SS}(F_0) \cap \dot{T}^* M), \end{cases}
$$

where the last equality follows from (4.3) applied to Φ and $\{\varphi_t^{-1}\}_{t\in I}$.

4.1 Non-displaceability: homogeneous case

We consider a C^1-map $\psi\colon M \to \mathbb{R}$ and we assume that

$$
(4.5)\qquad \text{the differential } d\psi(x) \text{ never vanishes.}
$$

Hence the section of T^*M defined by $d\psi$

$$
(4.6)\qquad \Lambda_\psi := \{(x; d\psi(x));\ x \in M\}
$$

is contained in $\dot{T}^*M$.

Theorem 4.1. *We consider $\Phi = \{\varphi_t\}_{t\in I}$ satisfying* (3.1)*, $\psi\colon M \to \mathbb{R}$ satisfying* (4.5) *and $F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support. We assume $\mathrm{R}\Gamma(M; F_0) \neq 0$. Then for any $t \in I$, $\varphi_t(\mathrm{SS}(F_0) \cap \dot{T}^*M) \cap \Lambda_\psi \neq \emptyset$.*

Proof. Let F, F_t be as in (4.2). Then F_t has compact support and $\mathrm{R}\Gamma(M; F_t) \neq 0$ by Corollary 1.7. Since $\mathrm{SS}(F_t) \subset \varphi_t(\mathrm{SS}(F_0) \cap \dot{T}^*M) \cup T^*_M M$, the result follows from Corollary 1.9. Q.E.D.

Corollary 4.2. *Let $\Phi = \{\varphi_t\}_{t\in I}$ and $\psi\colon M \to \mathbb{R}$ be as in Theorem 4.1. Let N be a non-empty compact submanifold of M. Then for any $t \in I$, $\varphi_t(\dot{T}^*_N M) \cap \Lambda_\psi \neq \emptyset$.*

4.2 Non-displaceability: Morse inequalities

In this subsection and in subsection 4.4 below we assume that $\mathbf{k}$ is a field. Let $F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and set

$$S_0 = \mathrm{SS}(F_0) \cap \dot{T}^*M.$$

Now we consider the hypotheses

(4.7) ψ is of class C^2 and the differential $d\psi(x)$ never vanishes,

(4.8) $\begin{cases} \text{there exists an open subset } S_{0,\mathrm{reg}} \text{ of } S_0 \text{ such that } S_{0,\mathrm{reg}} \text{ is a Lagrangian submanifold of class } \mathrm{C}^1 \text{ and } F_0 \text{ is a simple sheaf along } S_{0,\mathrm{reg}}. \end{cases}$

Lemma 4.3. *Let Λ be a smooth Lagrangian submanifold defined in a neighborhood of $p \in \dot{T}^*M$, let $G \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ and assume G is simple along Λ at p. Assume (4.7) and assume that Λ and Λ_ψ intersect transversally at p. Set $x_0 = \pi(p)$. Then*

$$\sum_j \dim H^j(\mathrm{R}\Gamma_{\{\psi(x)\geq\psi(x_0)\}}(G))_{x_0} = 1.$$

Proof. By the definition ([13, Definition 7.5.4]), $\mathrm{R}\Gamma_{\{\psi(x)\geq\psi(x_0)\}}(G)_{x_0}$ is concentrated in a single degree and its cohomology in this degree has rank one. Q.E.D.

In the sequel, for a finite set A, we denote by $\#A$ its cardinal.

Theorem 4.4. *We consider $\Phi = \{\varphi_t\}_{t\in I}$ satisfying (3.1), $\psi\colon M \to \mathbb{R}$ satisfying (4.7) and $F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support. We also assume (4.8). Let $t_0 \in I$. Assume that $\Lambda_\psi \cap \varphi_{t_0}(S_0)$ is contained in $\Lambda_\psi \cap \varphi_{t_0}(S_{0,\mathrm{reg}})$ and the intersection is finite and transversal. Then*

$$\#\big(\varphi_{t_0}(S_0) \cap \Lambda_\psi\big) \geq \sum_j b_j(F_0).$$

Proof. It follows from Corollary 1.7 that $b_j(F_t) = b_j(F_0)$ for all $j \in \mathbb{Z}$ and all $t \in I$.

Let $\{q_1, \dots, q_L\} = \Lambda_\psi \cap \varphi_{t_0}(\Lambda_0)$, $y_i = \pi(q_i)$ and set

$$W_i := \mathrm{R}\Gamma_{\{\psi(x)\geq\psi(y_i)\}}(F_{t_0})_{y_i}.$$

By Lemma 4.3, W_i is a bounded complex with finite-dimensional cohomologies and it follows from the Morse inequalities (1.9) that

$$\sum_j b_j(F_{t_0}) \leq \sum_j \sum_i b_j(W_i).$$

Moreover

$$\sum_j \dim H^j(\mathrm{R}\Gamma_{\{\psi(x)\geq\psi(y_i)\}}(F_{t_0})_{y_i}) = 1 \quad \text{for any } i,$$

and it implies

$$\sum_j \sum_i b_j(W_i) = \#\big(\mathrm{SS}(F_{t_0}) \cap \Lambda_\psi\big) = \#\big(\varphi_{t_0}(\mathrm{SS}(F_0) \cap \dot{T}^*M) \cap \Lambda_\psi\big).$$

Q.E.D.

Corollary 4.5. *Let $\Phi = \{\varphi_t\}_{t\in I}$, let $\psi\colon M \to \mathbb{R}$ satisfying (4.7) and let N be a compact submanifold of M. Let $t_0 \in I$. Assume that $\varphi_{t_0}(\dot{T}^*_N M)$ and Λ_ψ intersect transversally. Then*

$$\#(\varphi_{t_0}(\dot{T}^*_N M) \cap \Lambda_\psi) \geq \sum_j \dim H^j(N; \mathbf{k}_N).$$

Proof. Apply Theorem 4.4 with $F_0 = \mathbf{k}_N$. Q.E.D.

Remark 4.6. Corollaries 4.2 and 4.5 extend to the case where N is replaced with a compact submanifold with boundary or even with corners. In this case, one has to replace the conormal bundle $T^*_N M$ with the microsupport of the constant sheaf $\mathbf{k}_N$ on M. Note that this microsupport is easily calculated. For Morse inequalities on manifolds with boundaries, see the recent paper [16] and see also [15, 21] for related results.

4.3 Non-displaceability: non-negative Hamiltonian isotopies

Consider as above a manifold M and $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ a homogeneous Hamiltonian isotopy, that is, Φ satisfies (3.1). We define $f\colon \dot{T}^*M \times I \to \mathbb{R}$ homogeneous of degree 1 and $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ as in Lemma A.2. The following definition is due to [7] and is used in [4, 5] where the authors prove Corollary 4.14 below in the particular case where X and Y are points and other related results.

Definition 4.7. The isotopy Φ is said to be non-negative if $\langle \alpha_M, H_f \rangle \geq 0$.

Let eu_M be the Euler vector field on T^*M. Then $\langle \alpha_M, H_f \rangle = \mathrm{eu}_M(f)$ and since f is of degree 1 we have $\mathrm{eu}_M(f) = f$. Hence Φ is non-negative if and only if f is a non-negative valued function. We let (t, τ) be the coordinates on T^*I. Then by (A.4) this condition is also equivalent to

$$\Lambda \subset \{\tau \leq 0\}.$$

In order to prove Theorem 4.13 below, we shall give several results in sheaf theory.

Proposition 4.8. *Let N be a manifold, I an open interval of $\mathbb{R}$ containing 0. Let $F \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_{N\times I})$ and, for $t \in I$, set $F_t = F|_{N\times\{t\}} \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_N)$. Assume that*

(a) $\mathrm{SS}(F) \subset \{\tau \leq 0\}$,

(b) $\mathrm{SS}(F) \cap (T^*_N N \times T^*I) \subset T^*_{N\times I}(N \times I)$,

(c) $\mathrm{Supp}(F) \to I$ *is proper.*

Then we have:

(i) *for all $a \leq b$ in I there are natural morphisms $r_{b,a} \colon F_a \to F_b$,*

(ii) *$r_{b,a}$ induces a commutative diagram of isomorphisms*

$$\begin{tikzcd} \mathrm{R}\Gamma(N \times I; F) \arrow[d, "\wr"] \arrow[dr, "\sim"] & \\ \mathrm{R}\Gamma(N; F_a) \arrow[r, "\sim", "r_{b,a}"'] & \mathrm{R}\Gamma(N; F_b). \end{tikzcd}$$

Proof. By a similar argument to Corollary 1.7, (b) and (c) imply that $\mathrm{R}\Gamma(N\times I; F) \to \mathrm{R}\Gamma(N; F_t)$ is an isomorphism for any $t \in I$.

For $b \in I$ set $I_b = \{t \in I\,;\, t \leq b\}$ and $F' = F \otimes \mathbf{k}_{N\times I_b}$. Then F' also satisfies (a). Hence [13, Prop. 5.2.3] implies that $F' \simeq F' \circ \mathbf{k}_D$, where

$$\mathrm{D} = \{(s, t) \in I \times I\,;\, t \leq s\}\,.$$

We deduce the isomorphisms, for any $a \in I_b$:

$$F_a \simeq F' \circ \mathbf{k}_{\{a\}} \simeq F' \circ \mathbf{k}_{\mathrm{D}} \circ \mathbf{k}_{\{a\}} \simeq F' \circ \mathbf{k}_{[a,b]} \simeq F \circ \mathbf{k}_{[a,b]}. \tag{4.9}$$

The morphism $r_{b,a}$ is then induced by the morphism $\mathbf{k}_{[a,b]} \to \mathbf{k}_{\{b\}}$. Hence we obtain a commutative diagram

$$\begin{tikzcd}
 & \mathrm{R}\Gamma(N \times I; F) \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
\mathrm{R}\Gamma(N \times [a,b]; F) \arrow[rr] \arrow[d, "\wr"] & & \mathrm{R}\Gamma(N \times \{b\}\,; F) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma(N; F \circ \mathbf{k}_{[a,b]}) \arrow[rr] \arrow[d, "\wr"] & & \mathrm{R}\Gamma(N; F \circ \mathbf{k}_{\{b\}}) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma(N; F_a) \arrow[rr, "r_{b,a}"] & & \mathrm{R}\Gamma(N; F_b).
\end{tikzcd}$$

Q.E.D.

We recall that ω_X denotes the dualizing complex of a manifold X.

Lemma 4.9. *Let M be a manifold and X a locally closed subset of M. Let $i_X \colon X \to M$ be the embedding. We assume that the base ring $\mathbf{k}$ is not reduced to $\{0\}$.*

- (i) *Let $F \in \mathsf{D}(\mathbf{k}_M)$ and assume that there exists a morphism $u \colon F \to \mathrm{R}i_{X*}\mathbf{k}_X$ which induces an isomorphism $H^0(M; F) \xrightarrow{\sim} H^0(M; \mathrm{R}i_{X*}\mathbf{k}_X)$. Then $X \subset \mathrm{Supp}(F)$.*
- (ii) *Let $G \in \mathsf{D}(\mathbf{k}_M)$ and assume that there exists a morphism $v \colon i_{X!}\omega_X \to G$ which induces an isomorphism $H^0_c(M; i_{X!}\omega_X) \xrightarrow{\sim} H^0_c(M; G)$. Then $X \subset \mathrm{Supp}(G)$.*

Proof. (i) Let $x \in X$ and let $i_x \colon \{x\} \hookrightarrow M$ be the inclusion. For $x \in X$, the composition $\mathbf{k} \to H^0(M; \mathrm{R}i_{X*}\mathbf{k}_X) \to \mathbf{k}$ is the identity. Hence, in the commutative diagram

$$\begin{tikzcd}
H^0(M; F) \arrow[r, "\sim", "u"'] \arrow[d, "i_x^{-1}"'] & H^0(M; \mathrm{R}i_{X*}\mathbf{k}_X) \arrow[d, "i_x^{-1}"'] \\
H^0(F)_x \arrow[r] & \mathbf{k}
\end{tikzcd}$$

the map $H^0(F)_x \to \mathbf{k}$ is surjective. We conclude that $x \in \operatorname{Supp}(F)$.

(ii) For $x \in X$, the composition $H^0_{\{x\}}(M; i_{X!}\omega_X) \to H^0_c(M; i_{X!}\omega_X) \to \mathbf{k}$ is an isomorphism. Hence in the commutative diagram induced by v

$$\begin{tikzcd} H^0_{\{x\}}(M; i_{X!}\omega_X) \arrow[r] \arrow[d, tail, "a"'] & H^0_{\{x\}}(M; G) \arrow[d] \\ H^0_c(M; i_{X!}\omega_X) \arrow[r, "\sim", "b"'] & H^0_c(M; G) , \end{tikzcd}$$

the morphism a is injective and b is bijective. Hence $\mathbf{k} \simeq H^0_{\{x\}}(M; i_{X!}\omega_X) \to H^0_{\{x\}}(M; G)$ is injective. Therefore $H^0_{\{x\}}(M; G)$ does not vanish and $x \in \operatorname{Supp}(G)$. Q.E.D.

Lemma 4.10. *Let M be a non-compact connected manifold and let X be a compact connected submanifold of M. Then we have:*

(i) *The open subset $M \setminus X$ has at most two connected components.*

(ii) *Assume that there exists a relatively compact connected component U of $M \setminus X$. Then such a connected component is unique, X is a hypersurface and X coincides with the boundary of U.*

Proof. (i) We have an exact sequence

$$H^0(M; \mathbb{C}) \to H^0(M \setminus X; \mathbb{C}) \to H^1_X(M; \mathbb{C}).$$

The last term $H^1_X(M; \mathbb{C})$ is isomorphic to $H^0(X; H^1_X(\mathbb{C}_M))$. Since $H^1_X(\mathbb{C}_M)$ is locally isomorphic to $\mathbb{C}_X$ or 0, we have $\dim H^1_X(M; \mathbb{C}) \leq 1$. Hence we obtain $\dim H^0(M \setminus X; \mathbb{C}) \leq 2$. Hence $M \setminus X$ has at most two connected components.

(ii) Assume that there exists a relatively compact connected component U of $M \setminus X$. If $M \setminus X$ has another relatively compact connected component V, then $M = X \cup U \cup V$ by (i) and it is compact. It is a contradiction. Hence a relatively compact connected component U of $M \setminus X$ is unique if it exists. If X is not a hypersurface then $M \setminus X$ is connected and not relatively compact. It is a contradiction . Hence X is a hypersurface. Then it is obvious that X coincides with the boundary of U. Q.E.D.

Until the end of this subsection, we assume that $\Phi = \{\varphi_t\}_{t\in I}\colon \dot{T}^*M\times I \to \dot{T}^*M$ is a non-negative homogeneous Hamiltonian isotopy.

We define $g\colon \dot{T}^*M \times I \to \mathbb{R}$ by

$$g(p,t) = f(\varphi_t(p)^a, t) \quad (p \in \dot{T}^*M,\ t \in I). \tag{4.10}$$

Here $a\colon \dot{T}^*M \to \dot{T}^*M$ is the antipodal map.

Lemma 4.11. *Let Ψ be the symplectic isotopy given by $\Psi = \{a\circ\varphi_t^{-1}\circ a\}_{t\in I}$. Then we have $\dfrac{\partial\Psi}{\partial t} = H_{g_t}$, and Ψ is a non-negative Hamiltonian isotopy.*

Proof. Set $\psi_t = a\circ\varphi_t^{-1}\circ a$. Let Λ be the Lagrangian manifold associated to Φ as in Lemma A.1:

$$\Lambda \;=\; \Big\{\big(\varphi_t(v), v^a, t, -f(\varphi_t(v),t)\big)\,;\, v\in\dot{T}^*M, t\in I\Big\}.$$

Then we have

$$\begin{aligned}\Lambda \;&=\; \Big\{\big(w, \varphi_t^{-1}(w)^a, t, -f(w,t)\big)\,;\, w\in\dot{T}^*M, t\in I\Big\}\\ &=\; \Big\{\big(w^a, \psi_t(w), t, -f(w^a,t)\big)\,;\, w\in\dot{T}^*M, t\in I\Big\}.\end{aligned}$$

Since $f(w^a,t) = g(\varphi_t^{-1}(w^a)^a, t)$, the set

$$\Big\{\big(w^a, \psi_t(w), t, -g(\psi_t(w),t)\big)\,;\, w\in\dot{T}^*M, t\in I\Big\}$$

is Lagrangian. Hence Lemma A.1 implies that $\partial\Psi/\partial t = g_t$. The non-negativity of Ψ is obvious since g itself is non-negative. Q.E.D.

Lemma 4.12. *Let Λ_1 and Λ_2 be conic closed Lagrangian submanifolds of $\dot{T}^*M$. If either $\varphi_t(\Lambda_1)\subset\Lambda_2$ for all $t\in[0,1]$, or if $\Lambda_1\subset\varphi_t(\Lambda_2)$ for all $t\in[0,1]$, then $\varphi_t|_{\Lambda_1} = \mathrm{id}_{\Lambda_1}$ for all $t\in[0,1]$.*

Proof. (i) Let us treat the case where $\varphi_t(\Lambda_1)\subset\Lambda_2$ for all $t\in[0,1]$. We may assume that Λ_1 is connected. Then $\varphi_t(\Lambda_1)$ is a connected component of Λ_2, hence does not depend on t. Therefore $\varphi_t(\Lambda_1) = \Lambda_1$ for all $t\in[0,1]$. The hypothesis implies that $H_{f_t} = \partial\Phi/\partial t$ is tangent to Λ_1 for all $t\in[0,1]$. By Lemma A.2, $f_t = \langle\alpha_M, H_{f_t}\rangle$. Since Λ_1 is conic Lagrangian, the Liouville form α_M vanishes on the tangent bundle of Λ_1 and we deduce that f is identically 0 on $\Lambda_1\times[0,1]$.

Since f_t is a non-negative function on $\dot{T}^*M$, all points of Λ_1 are minima of f. It follows that $d(f_t) = 0$ on Λ_1 for all $t \in [0,1]$. Hence H_{f_t} also vanishes on Λ_1 and therefore $\varphi_t|_{\Lambda_1} = \mathrm{id}_{\Lambda_1}$ for all $t \in [0,1]$.
(ii) Now assume that $\Lambda_1 \subset \varphi_t(\Lambda_2)$ for all $t \in [0,1]$. Set $\psi_t = a \circ \varphi_t^{-1} \circ a$. Then $\{\psi_t\}_{t\in I}$ is a non-negative Hamiltonian isotopy by Lemma 4.11, and $\psi_t(\Lambda_1^a) \subset \Lambda_2^a$ holds for any $t \in [0,1]$. Hence step (i) implies that $\psi_t|_{\Lambda_1^a} = \mathrm{id}_{\Lambda_1^a}$.
Q.E.D.

Theorem 4.13. *Let M be a connected and non-compact manifold and let X, Y be two compact connected submanifolds of M. Let $\Phi = \{\varphi_t\}_{t\in I} \colon \dot{T}^*M \times I \to \dot{T}^*M$ be a non-negative homogeneous Hamiltonian isotopy. Assume that $[0,1] \subset I$ and $\varphi_1(\dot{T}^*_X M) = \dot{T}^*_Y M$. Then $X = Y$ and $\varphi_t|_{\dot{T}^*_X M} = \mathrm{id}_{\dot{T}^*_X M}$ for all $t \in [0,1]$.*[1]

Proof. By Lemma 4.12 it is enough to prove that $X = Y$ and $\varphi_t(\dot{T}^*_X M) \subset \dot{T}^*_X M$ for all $t \in [0,1]$.

We will distinguish two cases (see Lemma 4.10), respectively treated in (ii) and (iii) below:

(a) $M \setminus X$ or $M \setminus Y$ has no relatively compact connected component,

(b) both X and Y are the boundaries of relatively compact connected open subsets U and V of M, respectively.

(i) Let $K \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ be the quantization of Φ on I given by Theorem 3.7. By Proposition 3.2 (ii), the convolution with K_1 gives an equivalence of categories

$$\mathsf{D}^{\mathrm{lb}}_{T^*_X M \cup T^*_M M}(\mathbf{k}_M) \xrightarrow[K_1 \circ \cdot]{\sim} \mathsf{D}^{\mathrm{lb}}_{T^*_Y M \cup T^*_M M}(\mathbf{k}_M).$$

Moreover $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$, so that $\mathrm{SS}(K) \subset \{\tau \leq 0\}$. We consider $F_0 \in \mathsf{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support. We set:

$$F = K \circ F_0, \quad F_{t_0} = F \circ \mathbf{k}_{\{t=t_0\}}\,(t_0 \in I).$$

Then F satisfies (4.3) and we have $\mathrm{SS}(F) \subset \{\tau \leq 0\}$. Hence we may apply Proposition 4.8 and we deduce that, for all $a, b \in I$ with $a \leq b$, there are

[1]In an earlier draft of this paper, we only proved the first part of the conclusion of Theorem 4.13, namely that $X = Y$. We thank Stephan Nemirovski who asked us the question whether $\varphi_t|_{\dot{T}^*_X M}$ is the identity of $\dot{T}^*_X M$ for all $t \in [0,1]$.

natural morphisms $r_{b,a}\colon F_a \to F_b$ which induce isomorphisms $\mathrm{R}\Gamma(M; F_a) \xrightarrow{\sim} \mathrm{R}\Gamma(M; F_b)$.

(ii) Let us assume hypothesis (a). By Lemma 4.11, replacing φ_t with $a \circ \varphi_t^{-1} \circ a$ and X with Y if necessary, we may assume that any of the connected components of $M \setminus X$ is not relatively compact.

(ii-a) Let us show that $X = Y$. There exists $F_0 \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ such that $F_1 \simeq \mathbf{k}_Y$. We have $F_0 \simeq K_1^{-1} \circ \mathbf{k}_Y$ so that F_0 has compact support. We have also $\mathrm{SS}(F_t) \cap \dot{T}^*M = \varphi_t(\dot{T}^*_X M)$. Since $\mathrm{SS}(F_0) \subset T^*_X M \cup T^*_M M$, F_0 is locally constant outside X. Since $M \setminus X$ has no compact connected component, we deduce $\mathrm{Supp}(F_0) \subset X$. Hence by Lemma 4.9 (i), we have $Y \subset \mathrm{Supp}(F_0) \subset X$.

Since $M \setminus X \subset M \setminus Y$ and $M \setminus X$ has no relatively compact connected component, $M \setminus Y$ has also no relatively compact connected component. Hence by interchanging X and Y with the use of Lemma 4.11, we obtain $X \subset Y$. Thus we obtain $X = Y$.

(ii-b) Let us show $\dot{T}^*_X M \subset \varphi_t(\dot{T}^*_X M)$. Assuming that $p \in \dot{T}^*_X M \setminus \varphi_t(\dot{T}^*_X M)$, let us derive a contradiction. Take a C^1-function g such that $p = (x; dg(x))$ and $g|_X = 0$. Since $\mathrm{Supp}(F_0) \cap \{g < 0\} = \emptyset$, we obtain $H^0_{\{g<0\}}(F_0)_x \simeq 0$. Since $dg(x) \notin \mathrm{SS}(F_t)$, the morphism $H^0(F_t)_x \to H^0_{\{g<0\}}(F_t)_x$ is an isomorphism. Then we have a commutative diagram

$$\begin{tikzcd}
 & H^0(M;F_0) \arrow[r, "\sim"] \arrow[d] & H^0(M;F_t) \arrow[r, "\sim"] \arrow[d] & H^0(M;\mathbf{k}_Y) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{k} \\
 & H^0(F_0)_x \arrow[r] \arrow[d] & H^0(F_t)_x \arrow[r] \arrow[d, "\wr"] & (\mathbf{k}_Y)_x & \\
0 \arrow[r, "\sim"] & H^0_{\{g<0\}}(F_0)_x \arrow[r] & H^0_{\{g<0\}}(F_t)_x & &
\end{tikzcd}$$

Hence $\mathbf{k} \simeq H^0(M; F_t) \to H^0_{\{g<0\}}(F_t)_x$ is a monomorphism and also the zero morphism. This is a contradiction. Thus we obtain the desired result $\dot{T}^*_X M \subset \varphi_t(\dot{T}^*_X M)$. Thanks to Lemma 4.12, this completes the proof of the theorem under hypothesis (a).

(iii) Now we assume hypothesis (b). In this case, X and Y are hypersurfaces of M. Let $\dot{T}^{*,\mathrm{in}}_Y M$ be the "inner" conormal of Y, so that $\mathrm{SS}(\mathbf{k}_{\overline{V}}) = \overline{V} \cup \dot{T}^{*,\mathrm{in}}_Y M$ (see Example 1.2).

(iii-a) Let us first prove that $U = V$. As in (ii-a) there exists $F_0 \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ with compact support such that $F_1 \simeq \mathbf{k}_{\overline{V}}$. As in (ii-a) we see that $\mathrm{Supp}(F_0) \subset \overline{U} = U \cup X$. Part (i) gives a morphism $r_{1,0} \colon F_0 \to \mathbf{k}_{\overline{V}}$ which induces $H^0(M; F_0) \xrightarrow{\sim} H^0(M; \mathbf{k}_{\overline{V}}) \xrightarrow{\sim} \mathbf{k}$. Hence Lemma 4.9 implies that $\overline{V} \subset \mathrm{Supp}(F_0) \subset \overline{U}$. Then Lemma 4.11 implies the reverse inclusion. Hence $U = V$ and $X = Y$.

(iii-b) Let us prove that $\dot{T}^{*,\mathrm{in}}_X M \subset \varphi_t \varphi_1^{-1}(\dot{T}^{*,\mathrm{in}}_X M)$ for all $t \in [0,1]$. The proof is similar to the one in (ii-b). Assuming there exist $t \in [0,1]$ and $p \in (\dot{T}^{*,\mathrm{in}}_X M) \setminus \varphi_t \varphi_1^{-1}(\dot{T}^{*,\mathrm{in}}_X M)$, we shall derive a contradiction. Write $p = (x; dg(x))$ for a C^1-function g such that $g|_X = 0$. Hence $\{g > 0\}$ coincides with U on a neighborhood of x. Then we have a commutative diagram

$$
\begin{tikzcd}
 & H^0(M;F_0) \arrow[r, "\sim"] \arrow[d] & H^0(M;F_t) \arrow[r, "\sim"] \arrow[d] & H^0(M;\mathbf{k}_{\overline{V}}) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{k} \\
 & H^0(F_0)_x \arrow[r] \arrow[d] & H^0(F_t)_x \arrow[r] \arrow[d, "\wr"] & (\mathbf{k}_{\overline{V}})_x & \\
0 \arrow[r, "\sim"] & H^0_{\{g<0\}}(F_0)_x \arrow[r] & H^0_{\{g<0\}}(F_t)_x & &
\end{tikzcd}
$$

Hence $\mathbf{k} \simeq H^0(M; F_t) \to H^0_{\{g<0\}}(F_t)_x$ is a monomorphism and equal to the zero morphism. This is a contradiction. Hence we obtain $\dot{T}^{*,\mathrm{in}}_X M \subset \varphi_t \varphi_1^{-1}(\dot{T}^{*,\mathrm{in}}_X M)$, or equivalently $\varphi_t^{-1}(\dot{T}^{*,\mathrm{in}}_X M) \subset \varphi_1^{-1}(\dot{T}^{*,\mathrm{in}}_X M)$. Hence Lemma 4.12 implies that $\varphi_t^{-1}|_{\dot{T}^{*,\mathrm{in}}_X M} = \mathrm{id}_{\dot{T}^{*,\mathrm{in}}_X M}$, or $\varphi_t|_{\dot{T}^{*,\mathrm{in}}_X M} = \mathrm{id}_{\dot{T}^{*,\mathrm{in}}_X M}$. Lemma 4.11 permits us to apply this to $a \circ \varphi_t \circ a$, and we obtain $\varphi_t|_{a\dot{T}^{*,\mathrm{in}}_X M} = \mathrm{id}_{a\dot{T}^{*,\mathrm{in}}_X M}$. Thus we obtain $\varphi_t|_{\dot{T}^*_X M} = \mathrm{id}_{\dot{T}^*_X M}$. Q.E.D.

Corollary 4.14. *Let M be a connected manifold such that the universal covering $\widetilde{M}$ of M is non-compact. Let X and Y be simply connected and compact submanifolds of M with codimension ≥ 2. Let $\Phi \colon \dot{T}^*M \times I \to \dot{T}^*M$ be a non-negative homogeneous Hamiltonian isotopy such that $[0,1] \subset I$ and $\varphi_1(\dot{T}^*_X M) = \dot{T}^*_Y M$. Then $X = Y$ and $\varphi_t|_{\dot{T}^*_X M} = \mathrm{id}_{\dot{T}^*_X M}$ for all $t \in [0,1]$.*

Proof. Let $q \colon \widetilde{M} \to M$ and $p \colon \dot{T}^*\widetilde{M} \to \dot{T}^*M$ be the canonical projections. Let $f \colon \dot{T}^*M \times I \to \mathbb{R}$ be as in Lemma A.2 for Φ and set $\tilde{f} := f \circ (p \times \mathrm{id}_I) \colon \dot{T}^*\widetilde{M} \times I \to \mathbb{R}$. Let $\tilde{\Phi} \colon \dot{T}^*\widetilde{M} \times I \to \dot{T}^*\widetilde{M}$ the non-negative homogeneous Hamiltonian isotopy associated with $\tilde{f}$. We set $\tilde{\varphi}_t := \tilde{\Phi}|_{\dot{T}^*\widetilde{M} \times \{t\}} \colon \dot{T}^*\widetilde{M} \to$

$\dot{T}^*\widetilde{M}$. Then $p \circ \tilde{\varphi}_t = \varphi_t \circ p$. Let $q^{-1}(X) = \bigsqcup_{j\in J} \tilde{X}_j$ and $q^{-1}(Y) = \bigsqcup_{k\in K} \tilde{Y}_k$ be the decompositions into connected components. Then by the assumption, $\tilde{X}_j \to X$ and $\tilde{Y}_k \to Y$ are isomorphisms, and hence $\tilde{X}_j$ and $\tilde{Y}_k$ are connected and compact. Since $p^{-1}(\dot{T}^*_X M) = \sqcup_j \dot{T}^*_{\tilde{X}_j}\widetilde{M}$, we have

$$\bigsqcup_{j\in J} \tilde{\varphi}_1(\dot{T}^*_{\tilde{X}_j}\widetilde{M}) = \bigsqcup_{k\in K} \dot{T}^*_{\tilde{Y}_k}\widetilde{M}.$$

Since $\operatorname{codim}\tilde{X}_j$, $\operatorname{codim}\tilde{Y}_k > 1$, the unions are decompositions into connected components. So, for a given $j \in J$, there exists $k \in K$ such that $\tilde{\varphi}_1(\dot{T}^*_{\tilde{X}_j}\widetilde{M}) = \dot{T}^*_{\tilde{Y}_k}\widetilde{M}$. Hence Theorem 4.13 implies $\tilde{X}_j = \tilde{Y}_k$ and $\tilde{\varphi}_t|_{\dot{T}^*_{\tilde{X}_j}\widetilde{M}} = \mathrm{id}_{\dot{T}^*_{\tilde{X}_j}\widetilde{M}}$ for all $t \in [0,1]$. Finally we conclude $X = q(\tilde{X}_j) = q(\tilde{Y}_k) = Y$ and $\varphi_t|_{\dot{T}^*_X M} = \mathrm{id}_{\dot{T}^*_X M}$ for all $t \in [0,1]$. Q.E.D.

4.4 Non-displaceability: symplectic case

In this section we assume that $\mathbf{k}$ is a field. Using Theorems 4.1 and 4.4 we recover a well-known result solving a conjecture by Arnold [1, 6, 8, 10, 17]. We first state an easy geometric lemma.

Lemma 4.15. *Let $p\colon E \to X$ be a smooth morphism, let A, B be submanifolds of X and A' a submanifold of E. We assume that p induces a diffeomorphism $p|_{A'}\colon A' \xrightarrow{\sim} A$. We set $B' = p^{-1}(B)$. Then*

(4.11) *p induces a bijection $A' \cap B' \xrightarrow{\sim} A \cap B$,*

(4.12) *A' and B' intersect transversally if and only if A and B intersect transversally.*

Theorem 4.16. *Let N be a non-empty compact manifold. Let $\Phi\colon T^*N \times I \to T^*N$ be a Hamiltonian isotopy and assume that there exists a compact set $C \subset T^*N$ such that $\Phi|_{(T^*N\setminus C)\times I}$ is the projection on the first factor. We let $c = \sum_j \dim H^j(N; \mathbf{k}_N)$, the sum of the Betti numbers of N. Then for any $t \in I$ the intersection $\varphi_t(T^*_N N) \cap T^*_N N$ is never empty. Moreover its cardinality is at least c whenever the intersection is transversal.*

Proof. (i) We set $M = N \times \mathbb{R}$ and identify N with $N \times \{0\}$. We let $\widetilde{\Phi}\colon \dot{T}^*M \times I \to \dot{T}^*M$ be the homogeneous Hamiltonian isotopy given by Proposition A.6 and we set $\widetilde{\varphi}_t = \widetilde{\Phi}(\cdot, t)$.

We apply Theorems 4.1 and 4.4 to M, $\widetilde{\Phi}$, $F_0 = \mathbf{k}_N$ and $\psi = t$, the projection from M to $\mathbb{R}$. We obtain that the intersection $\widetilde{\varphi}_t(\dot{T}^*_N M) \cap \Lambda_\psi$ is a non-empty set whose cardinality is at least c whenever the intersection is transversal.

(ii) Now we compare $\widetilde{\varphi}_t(\dot{T}^*_N M) \cap \Lambda_\psi$ with the intersection considered in the theorem.

(ii-a) We apply Lemma 4.15 with $X = T^*N$, $E = T^*N \times \mathbb{R}^\times$, $p(x, \xi, \sigma) = (x, \xi/\sigma)$, $A = T^*_N N$, $B = \varphi_t(T^*_N N)$ and $A' = T^*_N N \times \{1\} \subset E$. We set $\Sigma_t := B' = p^{-1}(B)$. We have

$$\Sigma_t = \{(\sigma \cdot \varphi_t(x, 0), \sigma) \in T^*N \times \mathbb{R}^\times;\ x \in N, \sigma \in \mathbb{R}^\times\}. \tag{4.13}$$

By Lemma 4.15, $(T^*_N N \times \{1\}) \cap \Sigma_t \xrightarrow{\sim} T^*_N N \cap \varphi_t(T^*_N N)$ and one of these intersections is transversal if and only if the other one is.

(ii-b) We apply Lemma 4.15 with $X = T^*N \times \mathbb{R}^\times$, $E = T^*N \times \dot{T}^*\mathbb{R}$, $p(x, \xi, s, \sigma) = (x, \xi, \sigma)$, $A = \Sigma_t$, $B = T^*_N N \times \{1\}$ and $A' = \widetilde{\varphi}_t(\dot{T}^*_N M)$. We must check that the restriction of p to $\widetilde{\varphi}_t(\dot{T}^*_N M)$ induces an isomorphism $\widetilde{\varphi}_t(\dot{T}^*_N M) \xrightarrow{\sim} \Sigma_t$. This follows from (4.13) and the identity (see (A.8)):

$$\widetilde{\varphi}_t(\dot{T}^*_N M) = \{(\sigma \cdot \varphi_t(x, 0), u(x, 0, t), \sigma); x \in N, \sigma \in \mathbb{R}^\times\}.$$

We see easily that $B' = p^{-1}(B)$ is Λ_ψ. Hence Lemma 4.15 implies

$$\widetilde{\varphi}_t(\dot{T}^*_N M) \cap \Lambda_\psi \xrightarrow{\sim} \Sigma_t \cap (T^*_N N \times \{1\})$$

and one of these intersections is transversal if and only if the other one is. Together with (ii-a) and (i) this gives the theorem. Q.E.D.

A Appendix: Hamiltonian isotopies

We first recall some notions of symplectic geometry. Let $\mathfrak{X}$ be a symplectic manifold with symplectic form ω. We denote by $\mathfrak{X}^a$ the same manifold endowed with the symplectic form $-\omega$. The symplectic structure induces the Hamiltonian isomorphism $\mathbf{h}\colon T\mathfrak{X} \xrightarrow{\sim} T^*\mathfrak{X}$ by $\mathbf{h}(v) = \iota_v(\omega)$, where ι_v denotes the contraction with v. To a vector field v on $\mathfrak{X}$ we associate in this way a 1-form $\mathbf{h}(v)$ on $\mathfrak{X}$. For a C^∞-function $f\colon \mathfrak{X} \to \mathbb{R}$ the Hamiltonian vector field of f is by definition $H_f := -\mathbf{h}^{-1}(df)$.

The vector field v is called symplectic if its flow preserves ω. This is equivalent to $\mathscr{L}_v(\omega) = 0$ where $\mathscr{L}_v$ denotes the Lie derivative of v. By

Cartan's formula ($\mathscr{L}_v = d\,\iota_v + \iota_v\, d$) this is again equivalent to $d(\mathbf{h}(v)) = 0$ (recall that $d\omega = 0$). The vector field v is called Hamiltonian if $\mathbf{h}(v)$ is exact, or equivalently $v = H_f$ for some function f on $\mathfrak{X}$.

In this section we consider an open interval I of $\mathbb{R}$ containing the origin. We will use the following general notation: for a map $u\colon X \times I \to Y$ and $t \in I$ we let $u_t\colon X \to Y$ be the map $x \mapsto u(x,t)$.

A.1 Families of symplectic isomorphisms

Let $\Phi\colon \mathfrak{X}\times I \to \mathfrak{X}$ be a C^∞-map such that $\varphi_t{:=}\Phi(\cdot,t)\colon \mathfrak{X} \to \mathfrak{X}$ is a symplectic isomorphism for each $t \in I$ and is the identity for $t = 0$. The map Φ induces a time dependent vector field on $\mathfrak{X}$

$$v_\Phi := \frac{\partial \Phi}{\partial t}\colon \mathfrak{X} \times I \to T\mathfrak{X}. \tag{A.1}$$

Since $\varphi_t^*(\omega) = \omega$ we obtain by derivation $\mathscr{L}_{(v_\Phi)_t}(\omega) = 0$ for any $t \in I$, that is, $(v_\Phi)_t$ is a symplectic vector field. So the corresponding "time dependent" 1-form $\beta = \mathbf{h}(v_\Phi)\colon \mathfrak{X} \times I \to T^*\mathfrak{X}$ satisfies $d(\beta_t) = 0$ for any $t \in I$. The map Φ is called a Hamiltonian isotopy if $(v_\Phi)_t$ is Hamiltonian, that is, if β_t is exact for any t. In this case, integrating the 1-form β (which is C^∞ with respect to the parameter t) we obtain a C^∞-function $f\colon \mathfrak{X}\times I \to \mathbb{R}$ such that $\beta_t = -d(f_t)$. Hence we have

$$\frac{\partial \Phi}{\partial t} = H_{f_t}. \tag{A.2}$$

The fact that the isotopy Φ is Hamiltonian can be interpreted as a geometric property of its graph as follows. For a given $t \in I$ we let Λ_t be the graph of φ_t^{-1} and we let Λ' be the family of Λ_t's:

$$\begin{aligned}
\Lambda_t &= \{(\varphi_t(v), v)\,;\, v \in \mathfrak{X}^a\} \subset \mathfrak{X} \times \mathfrak{X}^a,\\
\Lambda' &= \{(\varphi_t(v), v, t)\,;\, v \in \mathfrak{X}^a,\ t \in I\} \subset \mathfrak{X} \times \mathfrak{X}^a \times I.
\end{aligned}$$

Then Λ_t is a Lagrangian submanifold of $\mathfrak{X} \times \mathfrak{X}^a$ and we ask whether we can lift Λ' as a Lagrangian submanifold Λ of $\mathfrak{X} \times \mathfrak{X}^a \times T^*I$ so that

$$(\mathrm{id}_{\mathfrak{X}\times\mathfrak{X}^a} \times \pi_I)|_\Lambda\colon \Lambda \xrightarrow{\sim} \Lambda'. \tag{A.3}$$

Lemma A.1. *We consider a* C^∞*-map* $\Phi\colon \mathfrak{X} \times I \to \mathfrak{X}$ *such that* φ_t *is a symplectic isomorphism for each* $t \in I$ *and we use the above notations. Then there exists a Lagrangian submanifold* $\Lambda \subset \mathfrak{X} \times \mathfrak{X}^a \times T^*I$ *satisfying* (A.3) *if and only if* Φ *is a Hamiltonian isotopy. In this case the possible* Λ *can be written*

$$\Lambda \;=\; \left\{\big(\Phi(v,t), v, t, -f(\Phi(v,t),t)\big)\,;\, v \in \mathfrak{X}, t \in I\right\}, \tag{A.4}$$

where the function $f\colon \mathfrak{X} \times I \to \mathbb{R}$ *is defined by* $(v_\Phi)_t = H_{f_t}$ *up to addition of a function depending only on* t.

If Λ exists we also have, extending notation (1.10) to the case where one manifold is not necessarily a cotangent bundle:

$$\Lambda_t = \Lambda \circ T^*_t I.$$

Proof. We write $T^*I \simeq I \times \mathbb{R}$. A manifold Λ satisfying (A.3) is written

$$\Lambda = \left\{\big(\Phi(v,t), v, t, \tau(v,t)\big)\,;\, v \in \mathfrak{X}^a, t \in I\right\}$$

for some function $\tau\colon \mathfrak{X}^a \times I \to \mathbb{R}$. Let us write down the condition that Λ be Lagrangian. For a given $(v,t) \in \mathfrak{X}^a \times I$ and $p = (\Phi(v,t), v, t, \tau(v,t)) \in \Lambda$ the tangent space $T_p\Lambda$ is generated by the vectors

$$\theta_0 = ((v_\Phi)_t, 0, 1, \frac{\partial \tau}{\partial t}) \quad \text{and} \quad \theta_\nu = ((d\varphi_t)(\nu), \nu, 0, (d\tau_t)(\nu)),$$

where ν runs over $T_v\mathfrak{X}^a$. Since φ_t is a symplectic isomorphism the θ_ν's are mutually orthogonal for the symplectic structure of $\mathfrak{X} \times \mathfrak{X}^a \times T^*I$. Hence Λ is Lagrangian if and only if θ_0 and θ_ν also are orthogonal, which is written:

$$\begin{aligned} 0 &= \omega\big((v_\Phi)_t, (d\varphi_t)(\nu)\big) - (d\tau_t)(\nu) \\ &= \big(\mathbf{h}((v_\Phi)_t) - d(\tau_t \circ \varphi_t^{-1})\big)((d\varphi_t)(\nu)). \end{aligned}$$

This holds for all $\nu \in T_v\mathfrak{X}^a$ if and only if $\mathbf{h}((v_\Phi)_t) = d(\tau_t \circ \varphi_t^{-1})$, or equivalently $-H_{\tau_t \circ \varphi_t^{-1}} = (v_\Phi)_t$. Q.E.D.

Exact case We assume that the symplectic form ω is exact and write $\omega = d\alpha$. We consider $\Phi\colon \mathfrak{X} \times I \to \mathfrak{X}$ as above but now we ask that $\varphi_t^*(\alpha) = \alpha$ for all $t \in I$. Then it is well-known (see for example [18, Corollary 9.19]) that Φ is a Hamiltonian isotopy. More precisely v_Φ is the Hamiltonian vector field of

$$f = \langle \alpha, v_\Phi \rangle \colon \mathfrak{X} \times I \to \mathbb{R}. \tag{A.5}$$

Indeed the condition on φ_t implies by derivation $\mathscr{L}_{v_\Phi}(\alpha) = 0$. Hence Cartan's formula yields:

$$d(f_t) = \mathscr{L}_{(v_\Phi)_t}(\alpha) - \iota_{(v_\Phi)_t}(\omega) = -\iota_{(v_\Phi)_t}(\omega) = -\mathbf{h}((v_\Phi)_t).$$

This holds in particular when $\mathfrak{X} = \dot{T}^*M$ for some manifold M. We consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ such that

$$\begin{cases} \varphi_t \text{ is a homogeneous symplectic isomorphism for each } t \in I, \\ \varphi_0 = \mathrm{id}_{\dot{T}^*M}\,. \end{cases} \tag{A.6}$$

In this case the function f given in (A.5) is homogeneous of degree 1 in the fibers of $\dot{T}^*M$ and it is the only homogeneous function such that $(v_\Phi)_t = H_{f_t}$. So we have the first part of the following lemma.

Lemma A.2. *Let $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ satisfying* (3.1). *Then*

(i) *Φ is a Hamiltonian isotopy and there exists a unique conic Lagrangian submanifold Λ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ satisfying* (A.3)*: setting $f = \langle \alpha_M, \partial\Phi/\partial t \rangle$ we have*

$$\Lambda = \Big\{ \big(\Phi(x,\xi,t), (x,-\xi), (t, -f(\Phi(x,\xi,t),t))\big)\,;\, (x,\xi) \in \dot{T}^*M, t \in I \Big\},$$

(ii) *the set $\Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$ is closed in $T^*(M \times M \times I)$ and for any $t \in I$ the inclusion $i_t\colon M \times M \to M \times M \times I$ is non-characteristic for Λ and the graph of φ_t is $\Lambda_t = \Lambda \circ T^*_t I$.*

Proof. (i) is already proved.

(ii) In local homogeneous symplectic coordinates $(x, y; \xi, \eta) \in T^*(M \times M)$, $(t; \tau) \in T^*I$, the construction of Λ implies that for any compact set $C \subset M \times M \times I$ there exists $D > 0$ such that $|\tau| \le D|\xi|$, $|\xi| \le D|\eta|$ and $|\eta| \le D|\xi|$

for any $(x,y,t;\xi,\eta,\tau) \in \Lambda \cap \pi^{-1}_{M\times M\times I}(C)$. Hence the same inequalities hold on the closure $\overline{\Lambda}$ of Λ. Hence if $(x,y,t;\xi,\eta,\tau) \in \overline{\Lambda} \setminus (\dot{T}^*M \times \dot{T}^*M \times T^*I)$, then $\xi = \eta = 0$ and $\tau = 0$, and hence it belongs to the zero-section.

Hence $\Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$ is closed. We also have seen that Λ does not meet $T^*_{M\times M}(M \times M) \times \dot{T}^*I$ which is the non-characteristicity condition.

Q.E.D.

A.2 Families of conic Lagrangian submanifolds

Since the results in this section are well-known (they go back to Paulette Libermann), we state them without proofs. Note that we only use them in Corollary 3.13.

Definition A.3. Let M be a manifold and I an open interval containing 0. Let S_0 be a closed conic Lagrangian submanifold of $\dot{T}^*M$. A deformation of S_0 indexed by I is the data of a C^∞-map $\Psi\colon S_0 \times I \to \dot{T}^*M$ such that, setting $\psi_t := \Psi(\cdot, t)$ and $S_t = \psi_t(S_0)$, we have

(i) ψ_0 is the identity embedding,

(ii) ψ_t is homogeneous for the action of $\mathbb{R}_{>0}$ for each $t \in I$,

(iii) S_t is a closed conic Lagrangian submanifold of $\dot{T}^*M$ for each $t \in I$,

(iv) the map $S_0 \times I \to (\dot{T}^*M) \times I$, $(s,t) \mapsto (\Psi(s,t), t)$, is an embedding.

We let $S' = \{(s,t);\ t \in I, s \in S_t\} \subset (\dot{T}^*M) \times I$ be the image of the embedding in (iv). So it is a closed submanifold of $(\dot{T}^*M) \times I$. Note that ψ_t induces a diffeomorphism $\psi_t\colon S_0 \xrightarrow{\sim} S_t$ for each $t \in I$.

Lemma A.4. *Let S_0 be a closed conic Lagrangian submanifold of $\dot{T}^*M$ and let $\Psi\colon S_0 \times I \to \dot{T}^*M$ be a deformation of S_0 as above. Then there exists a unique closed conic Lagrangian submanifold $S \subset \dot{T}^*(M \times I)$ such that $\dot{T}^*(M \times I) \to T^*M \times I$ induces a diffeomorphism $S \xrightarrow{\sim} S'$.*

Moreover for any $t \in I$ the inclusion $i_t\colon M \to M \times I$ is non-characteristic for S and we have $S_t = S \circ T^*_t I$.

We remark that S, like S', only depends on the family of $\{S_t\}_t$, not on the parametrization Ψ.

For a deformation of a closed conic Lagrangian submanifold we consider a condition similar to (3.3).

$$\text{there exists a compact subset } A \text{ of } M \text{ such that for all } t \in I\text{:}$$

$$\begin{cases} \psi_t|_{S_0 \cap \dot\pi_M^{-1}(M \setminus A)} = \mathrm{id}_{S_0 \cap \dot\pi_M^{-1}(M \setminus A)}, \\ \psi_t(S_0 \cap \dot\pi_M^{-1}(A)) \subset \dot\pi_M^{-1}(A). \end{cases} \tag{A.7}$$

Proposition A.5. *Let S_0 be a closed conic Lagrangian submanifold of $\dot T^*M$ and let $\Psi \colon S_0 \times I \to \dot T^*M$ be a deformation of S_0 satisfying* (A.7)*.*

Then there exists $\Phi \colon \dot T^*M \times I \to \dot T^*M$ satisfying hypotheses (3.1) *and* (3.3) *such that*

$$\Phi|_{S_0 \times I} = \Psi.$$

A.3 Adding a variable

In this subsection we recall the link between non-homogeneous symplectic geometry and homogeneous symplectic geometry with an extra variable.

We denote by (s, σ) the coordinates on $T^*\mathbb{R}$ with σds as the Liouville form. For a manifold M we define the map

$$\rho = \rho_M \colon T^*M \times \dot T^*\mathbb{R} \to T^*M, \qquad (x, \xi, s, \sigma) \mapsto (x, \xi/\sigma).$$

We consider a Hamiltonian isotopy $\Phi \colon T^*M \times I \to T^*M$ as in Appendix A.1 but we do not assume that it is homogeneous. We shall show that Φ lifts to a homogeneous Hamiltonian isotopy of $T^*M \times \dot T^*\mathbb{R}$.

Let $f \colon T^*M \times I \to \mathbb{R}$ be a function such that $\partial\Phi/\partial t = H_{f_t}$ (see (A.2)). We set

$$\tilde f := (f \circ \rho) \cdot \sigma.$$

Then $\tilde f_t$ is a homogeneous function on $T^*M \times \dot T^*\mathbb{R}$ of degree 1.

Proposition A.6. *Let $\Phi \colon T^*M \times I \to T^*M$ be a Hamiltonian isotopy and let f and $\tilde f$ be as above.*

(i) *Then there exists a homogeneous Hamiltonian isotopy $\widetilde\Phi \colon (T^*M \times \dot T^*\mathbb{R}) \times I \to T^*M \times \dot T^*\mathbb{R}$ such that $\partial\widetilde\Phi/\partial t = H_{\tilde f_t}$ and the following diagram* *commutes:*

$$\begin{tikzcd}
T^*M \times \dot{T}^*\mathbb{R} \times I \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\rho\times\mathrm{id}_I"'] & T^*M \times \dot{T}^*\mathbb{R} \arrow[d, "\rho"] \\
T^*M \times I \arrow[r, "\Phi"] & T^*M.
\end{tikzcd}$$

Moreover there exists a C^∞*-function* $u\colon (T^*M)\times I \to \mathbb{R}$ *such that*

(A.8) $$\widetilde{\Phi}(x,\xi,s,\sigma,t) = (x',\xi',s+u(x,\xi/\sigma,t),\sigma),$$

where $(x',\xi'/\sigma) = \varphi_t(x,\xi/\sigma)$.

(ii) *We assume moreover that* M *is connected and* φ_t *is the identity outside a compact subset* $C \subset T^*M$. *Then* $\widetilde{\Phi}$ *extends to a homogeneous Hamiltonian isotopy* $\widetilde{\Phi}\colon \dot{T}^*(M\times\mathbb{R})\times I \to \dot{T}^*(M\times\mathbb{R})$ *such that*

(A.9) $$\widetilde{\Phi}(x,\xi,s,0,t) = (x,\xi,s+v(t),0),$$

for some C^∞*-function* $v\colon I\to\mathbb{R}$.

Proof. We have to describe the Hamiltonian vector field $H_{\tilde{f}}$ of $\widetilde{f}$. We denote by $p\colon T^*M\times\dot{T}^*\mathbb{R}\to\dot{T}^*\mathbb{R}$ the projection $(x,\xi,s,\sigma)\mapsto(s,\sigma)$. Then (ρ,p) defines an isomorphism

(A.10) $$\psi\colon T^*M\times\dot{T}^*\mathbb{R} \xrightarrow{\sim} T^*M\times\dot{T}^*\mathbb{R}.$$

For a point $q=(x,\xi,s,\sigma)\in T^*M\times\dot{T}^*\mathbb{R}$, ψ defines an isomorphism on the tangent spaces:

(A.11) $$d\psi = d\rho_q \times dp_q\colon T_q(T^*M\times T^*\mathbb{R}) \xrightarrow{\sim} T_{(x,\xi/\sigma)}(T^*M)\oplus T_{(s,\sigma)}(T^*\mathbb{R}).$$

Setting $\omega_M = d\alpha_M$, where α_M is the Liouville form on T^*M, we have

$$\begin{aligned}
\alpha_{M\times\mathbb{R}}|_{T^*M\times\dot{T}^*\mathbb{R}} &= \sigma\rho^*(\alpha_M)+p^*(\alpha_\mathbb{R}),\\
\omega_{M\times\mathbb{R}}|_{T^*M\times\dot{T}^*\mathbb{R}} &= \sigma\rho^*(\omega_M)+p^*(\omega_\mathbb{R})+d\sigma\wedge\rho^*(\alpha_M).
\end{aligned}$$

In the sequel we fix t and we set $\tilde{f}_t = \widetilde{f}(\cdot,t)$ and $f_t = f(\cdot,t)$. Then $H_{\tilde{f}_t}$ is determined by $\iota_{H_{\tilde{f}_t}}(\omega_{M\times\mathbb{R}}) = -d\tilde{f}_t$. We decompose $(H_{\tilde{f}_t})_q = v_M + v_\mathbb{R}$

according to (A.11) and we also use the decomposition of $T^*_q(T^*M \times T^*\mathbb{R})$ induced by (A.11). Then we find

$$\iota_{H_{\tilde{f}_t}}(\omega_{M\times\mathbb{R}}) = \big(\sigma\iota_{v_M}(\omega_M) + \langle v_{\mathbb{R}}, d\sigma\rangle\alpha_M\big) + \big(\iota_{v_{\mathbb{R}}}(\omega_{\mathbb{R}}) - \langle v_M, \alpha_M\rangle d\sigma\big).$$

Since $d\tilde{f}_t = \sigma\rho^* df_t + \rho^*(f_t)d\sigma$ we obtain

$$\begin{aligned} -df_t &= \iota_{v_M}(\omega_M) + \sigma^{-1}\langle v_{\mathbb{R}}, d\sigma\rangle\alpha_M, \\ -\rho^*(f_t)d\sigma &= \iota_{v_{\mathbb{R}}}(\omega_{\mathbb{R}}) - \langle v_M, \alpha_M\rangle d\sigma. \end{aligned}$$

The second equality gives $v_{\mathbb{R}} = a\frac{\partial}{\partial s}$ for some function a. Then we have $\langle v_{\mathbb{R}}, d\sigma\rangle = 0$, which implies $v_M = H_{f_t}$ by the first equality, and hence $a = (f_t - \langle H_{f_t}, \alpha_M\rangle) \circ \rho = (f_t - \mathrm{eu}_M(f_t)) \circ \rho$. Finally, letting $g := f - \mathrm{eu}_M(f)$ be a function on $T^*M \times I$, we obtain

$$\psi_*(H_{\tilde{f}_t}) = H_{f_t} + \rho^*(g_t)\frac{\partial}{\partial s}.$$

Let us define $u\colon T^*M \times I \to \mathbb{R}$ by the differential equation:

$$\text{(A.12)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} &= g_t \circ \varphi_t, \\ u|_{t=0} &= 0. \end{cases}$$

We define $\widetilde{\Phi}$ by (A.8). Then, we can see easily that

$$\frac{\partial\widetilde{\Phi}}{\partial t} = H_{\tilde{f}_t}.$$

Hence $\widetilde{\Phi}$ is the desired homogeneous Hamiltonian isotopy.

(ii) The functions f_t and g_t are constant functions outside C. Hence u_t is also a constant function outside C taking the value $v(t)$. Then $\widetilde{\Phi}$ extends to a homogeneous Hamiltonian isotopy $\widetilde{\Phi}\colon \dot{T}^*(M \times \mathbb{R}) \times I \to \dot{T}^*(M \times \mathbb{R})$ by (A.9). Q.E.D.

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